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Par

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Sujet :

# Processus (multi-)fractionnaires à paramètres multidimensionnels et régularité höldérienne

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## Acknowledgement/Remerciements


> *"...une bonne partie des mathématiques devenues utiles se sont développées sans aucun désir d'être utiles, dans une situation où personne ne pouvait savoir dans quels domaines elles deviendraient utiles. Il n'y avait aucune indication générale qu'elles deviendraient utiles. C'est vrai de toute la science."*
>
> John Von Neumann

Le poids des mathématiques dans ma vie ressemble déjà au cours d'un fleuve dont les méandres tortueux ne peuvent laisser prévoir les régions qu'il va visiter. Au fil de cette relation, tantôt passionnée tantôt délaissée, j'ai eu la chance de rencontrer des personnes dont l'influence est indissociable de la rédaction de ce mémoire de thèse.

Le passage d'une situation de cancre au collège à celle de passionné par une discipline plus synonyme de rigueur que de beauté artistique, a certainement eu lieu poussé par une volonté de m'extraire du monde réel pour me réfugier dans un univers d'abstraction où ressentir la beauté d'une démonstration procure une satisfaction comparable à celle qu'on éprouve ébloui par quelque tableau ou encore transporté par un concerto. Comment résister à l'envie d'évoquer le parallèle entre l'étude des mathématiques et la pratique de la montagne ? Le grand alpiniste Lionel Terray conclut le livre de sa vie, *Les conquérants de l'inutile*,

> *"Mais s'ils évoluent près du ciel, dans la pureté infinie d'un monde de lumière et de beauté, les alpinistes ne sont pas des anges. Ils sont toujours des hommes et leur coeur reste souillé par la noirceur du monde d'où ils viennent et où, tôt ou tard, ils devront retourner."*

L'initiation au monde des mathématiques fut le fruit de la rencontre de professeurs captivants, tels que M. Claes, Mme Laplace et M. François, qui ont guidé ma route jusqu'au bagne douillet du préparationaire dans cette pépinière d'esprits que constitue le Lycée Louis-le-Grand. J'ai l'impression d'y avoir vécu mes meilleures années. C'est finalement là-bas que l'enseignement de M. Delezoide et M. Chevallet, dans la pure tradition française influencée par Bourbaki, combla mes aspirations.

Mon entrée à l'Ecole Centrale marqua le début d'une longue séparation d'avec les mathématiques. Ne me destinant pas spécialement aux études d'ingénieur, je ressentis alors comme une insatisfaction qui ne me quitta plus pendant les trois premières années de ma vie professionnelle chez Dassault Aviation.

À cette époque, une réflexion sur mes satisfactions passées ramena les mathématiques à mon esprit. Ainsi, naquit l'idée de concourir à l'agrégation, sans toutefois l'intention de quitter mon emploi pour l'enseignement. Cette démarche me permit de prendre conscience du manque que j'avais, loin de la gymnastique intellectuelle imposée par l'abstraction. D'autre part, lors de notre rencontre à l'oral de l'agrégation, Pascal Massart me conseilla de tout faire pour concilier des études de mathématiques, à travers la poursuite d'un DEA suivi d'une thèse, avec mon travail d'ingénieur.

Je rendis alors visite à Bruno Stoufflet, directeur de la stratégie scientifique de Dassault Aviation, pour savoir dans quelle mesure il serait envisageable de reprendre des études universitaires dans le cadre d'une activité dans l'entreprise. Sans être totalement convaincu par ma démarche, celui-ci me conseilla de m'intéresser au calcul stochastique, branche des mathématiques non représentée dans son département d'Études Scientifiques Amont.

Ainsi, je décidai de suivre les cours du DEA de maths pures de l'université d'Orsay. Le choix des matières que j'allais étudier fut quelquechose de très difficile : j'avais une telle

soif d'apprendre que tout m'intéressait. Parmi une petite poignée de jeunes normaliens, je suivis alors les cours de *théorie analytique des nombres* du professeur Fouvry, ainsi que de *mouvement brownien et calcul stochastique* des professeurs Lejan et Werner. Le premier m'apporta un plaisir que j'avais oublié depuis mon séjour rue Saint-Jacques. Dès cette époque, la pureté et la gratuité de l'algèbre avaient été pour moi un objet de fascination. Le choix du deuxième cours était poussé par l'espoir de réunir le plaisir de l'abstraction mathématique et la résolution de problèmes concrets issus du monde de l'entreprise. La rédaction de mon mémoire de DEA sous l'encadrement de W. Werner, me fit prendre conscience que les objets emprumtés aux mathématiques appliquées que sont les probabilités, peuvent être sujets de questions dont la réponse prend sa source dans le monde des mathématiques pures. L'étude des exposants critiques du mouvement brownien plan en fut pour moi la magnifique illustration.

À la fin de cette année, je partis en quête d'un sujet de thèse à la fois motivant et utile pour la suite de ma jeune vie professionnelle. J'eus la chance d'être conseillé par les professeurs Werner et Massart, qui eurent la gentillesse de recevoir des gens de chez Dassault pour m'aider dans ma quête. Parmi elles, Yves Auffray, membre du département d'Études Scientifiques Amonts dans lequel je me trouve actuellement, ne ménageait pas sa peine pour m'aider à réaliser mon projet professionnel. Il réussit à extraire un, voire plusieurs sujets, de la problématique complexe du suivi de terrain par un avion de chasse.

Parallèlement, j'eus la chance d'être reçu par D. Verwaerde, conseiller en matière de mathématiques appliquées auprès du Haut Commissaire à l'Energie Atomique, par B. Giraud du Service de Physique Théorique du CEA, E. Klein physicien au CEA. Chacune de ces personnes me convainquirent de la nécessité d'entreprendre un travail de thèse en parallèle avec mon travail chez Dassault. Je contactai alors Jacques Lévy-Véhel, alors chef du projet Fractales de l'INRIA Rocquencourt. Celui-ci sembla intéressé par mon profil d'ingénieur animé de la volonté de faire de la recherche mathématique. Cette rencontre fut déterminante. Jacques personnifiait la capacité à abstraire les problèmes concrets du monde des ingénieurs. Celui-ci n'eut aucune peine à imaginer comment une extension dans $\mathbf{R}^2$ du *mouvement brownien multifractionnaire*, constituerait un bon candidat comme modèle aléatoire du terrain dans le cadre très appliqué du vol à très basse altitude. Sa vision ne tarda pas à convaincre Bruno Stoufflet qui m'accepta dans son département pour travailler sur ce sujet.

La littérature traitant des processus à plusieurs paramètres me semblait d'emblée très touffue. Wendelin Werner me conseilla alors de m'adresser à Davar Khoshnevisan qui écrivait un ouvrage sur le sujet. Davar eut la gentillesse de m'envoyer son livre non encore publié, dont la lecture m'éclaira beaucoup, non seulement sur ces processus indexés par $\mathbf{R}_+^N$, mais aussi sur certains points que j'avais mal compris concernant les processus indexés par $\mathbf{R}_+$.

La rencontre d'Ely Merzbach fut elle-aussi déterminante, lors d'une École d'été qu'il organisait en Italie sur les *processus spatiaux*. Le cadre général de la théorie des *set-indexed martingales* me donnait l'envie de définir une extension *set-indexed* du mBm. Dans cet élan, je lui proposais de regarder ce qu'il était possible de construire à partir du brownien fractionnaire.

Enfin, je voudrais consacrer ces dernières lignes de palabres, à remercier les illustres membres de mon jury, tout d'abord Davar Khoshnevisan et Yimin Xiao pour avoir accepté la tache, pas forcément agréable, de rapporteurs, et ensuite Ely Merzbach, Zhan Shi et Wendelin Werner dont la présence dans mon jury m'honore.

Je voulais ici remercier toutes ces personnes dont l'influence est indissociable du travail que je vais exposer. Peut-être que grâce à leur concours, je réussirai ce grand écart entre la pureté de l'inutilité et la satisfaction d'une réalisation concrête.

Bien entendu, je ne t'oublie pas, Jacques ! Tu m'as guidé pendant ces quelques années, ponctuées parfois de moments difficiles. Grâce à toi, j'espère réussir à prendre mon envol et parvenir à m'épanouir au travers de jolies réalisations mathématiques.

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# Chapitre I

# Introduction, synthèse des principaux résultats

> *“Monsieur Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels. Un philosophe tel que lui aurait dû savoir que le but unique de la Science, c'est l'honneur de l'esprit humain et que, sous ce titre, une question de nombres vaut bien une question de système du monde.”*
>
> Gustav Jacobi

## 1 Le mouvement brownien fractionnaire

Historiquement le *mouvement brownien* fut la dénomination du mouvement de particules de pollen à la surface de l'eau, observé par le botaniste R. Brown au début du 19ème siècle. Etudié par A. Einstein en 1905, ce mouvement aléatoire, résultant de l'interaction avec les molécules d'eau, possède les propriétés suivantes :

- les accroissements sont indépendants,
  i.e. quelle que soit la suite $t_1 < \cdots < t_n$ de $\mathbf{R}_+$, les variables $B_{t_i} - B_{t_{i-1}}$ sont indépendantes,
- les accroissements sont des variables aléatoires gaussiennes,
- le mouvement est continu.

Ces trois propriétés caractérisent le mouvement brownien, objet mathématique au centre de l'étude des processus stochastiques. Par ailleurs, il possède de nombreuses autres propriétés. Par exemple, il est à la fois une martingale, une diffusion et un processus de Markov. Son étude approfondie traitée dans [18], [24] et [25]. De plus,

- ses accroissements sont stationnaires

  $$\forall h \in \mathbf{R}_+; \quad \{B_{t+h} - B_h\}_{t\in\mathbf{R}_+} \stackrel{(d)}{=} \{B_t - B_0\}_{t\in\mathbf{R}_+}$$

  et

  $$\forall s,t \in \mathbf{R}_+; \quad E\left[B_t - B_s\right]^2 = |t-s|$$

- il est auto-similaire

  $$\forall a \in \mathbf{R}_+; \quad \{B_{at}\}_{t\in\mathbf{R}_+} \stackrel{(d)}{=} a^{\frac{1}{2}} \{B_t\}_{t\in\mathbf{R}_+}$$

Le *mouvement brownien fractionnaire* est défini comme une généralisation du mouvement brownien standard, pour lequel on "abandonne" l'indépendance des accroissements. Introduit par Kolmogorov en 1940 pour l'étude de la turbulence, puis par Mandelbrot et Van Ness en 1958, il apparaît assez efficace dans la modélisation de certains phénomènes tel que le traffic sur Internet ou les cours financiers. Très récemment, l'introduction du calcul stochastique par rapport au fBm a ouvert de nouvelles perspectives d'application.

Ainsi, on définit le mouvement brownien fractionnaire $B^H$ comme la modification continue du processus gaussien centré de covariance

$$\forall s,t \in \mathbf{R}_+; \quad E\left[B_s^H B_t^H\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right]$$

Le paramètre $H \in ]0,1[$ est parfois appelé paramètre de Hurst ou paramètre d'auto-similarité.

On peut voir aisément que

$$\forall s,t \in \mathbf{R}_+; \quad E\left[B_t^H - B_s^H\right]^2 = |t-s|^{2H}$$

Ainsi, le mouvement brownien standard est un mouvement brownien fractionnaire pour lequel $H = \frac{1}{2}$.

## 1.1 Les différentes représentations

Rappelons brièvement les principales représentations classiques du mouvement brownien fractionnaire, sur lesquelles reposent la définition du mouvement brownien multifractionnaire, ainsi que ses extensions à plusieurs paramètres. Toutes les représentations sont données à une constante multiplicative près.
Avant cela, rappelons la définition de l'intégrale par rapport au bruit blanc.

### Le bruit blanc et le processus isonormal

Comme l'application $\Sigma : \mathcal{B}(\mathbf{R}^N) \times \mathcal{B}(\mathbf{R}^N) \to \mathbf{R}_+$, où $\mathcal{B}(\mathbf{R}^N)$ est la tribu borélienne de $\mathbf{R}^N$, telle que

$$\forall U,V \in \mathcal{B}(\mathbf{R}^N); \quad \Sigma(U,V) = Leb(U \cap V) \tag{I.1}$$

est définie positive, on peut définir le *bruit blanc* sur $\mathbf{R}^N$ comme le processus gaussien centré $\mathbb{W} = \left\{\mathbb{W}(U);\ U \in \mathcal{B}(\mathbf{R}^N)\right\}$ de fonction de covariance $\Sigma$ (cf [19]).
Le bruit blanc vérifie les propriétés suivantes :

1. Pour tous boréliens $U$ et $V$ disjoints, les variables aléatoires $\mathbb{W}(U)$ et $\mathbb{W}(V)$ sont indépendantes,

2. Pour tous $U, V \in \mathcal{B}(\mathbf{R}^N)$,

$$\mathbb{W}(U \cup V) = \mathbb{W}(U) + \mathbb{W}(V) - \mathbb{W}(U \cap V) \quad \text{p.s.}$$

3. Soit $(U_i)_{i\in\mathbf{N}}$ une suite de boréliens disjoints tel que $\sum_{i=1}^{\infty} Leb(U_i) < \infty$. Alors on a presque surement

$$\mathbb{W}\left(\bigcup_{i=1}^{\infty} U_i\right) = \sum_{i=1}^{\infty} \mathbb{W}(U_i)$$

Ces propriétés suggèrent l'idée de construire une intégrale par rapport au bruit blanc. Pour cela, on considère habituellement les *fonctions élémentaires*, qui sont définies de la forme $f = \mathbb{1}_E$ où $E \in \mathcal{B}(\mathbf{R}^N)$ et $Leb(E) < \infty$, ainsi que leurs combinaisons linéaires finies, appelées *fonctions simples*.

On peut définir l'intégrale d'une fonction $f = \mathbb{1}_E$ par

$$\int f(s).\mathbb{W}(ds) = \mathbb{W}(E)$$

et étendre cette définition aux fonctions simples par linéarité, puis par densité à $L^2(\mathbf{R}^N)$. On obtient ainsi le processus isonormal.
Le *processus isonormal* $W = \{W(h);\ h \in L^2(\mathbf{R}^N)\}$ est un processus gaussien centré, indexé par $L^2(\mathbf{R}^N)$, tel que

$$\forall h_1, h_2 \in L^2(\mathbf{R}^N); \quad E\left[W(h_1).W(h_2)\right] = \langle h_1, h_2 \rangle$$

De plus, on a $\forall \alpha, \beta \in \mathbf{R}$, $\forall f, g \in L^2(\mathbf{R}^N)$,

$$W(\alpha.f + \beta.g) = \alpha.W(f) + \beta.W(g) \quad \text{p.s.}$$

Les détails de cette construction peuvent être consultés dans [19]. Le processus $B = \{\mathbb{W}([0,t];\ t \in \mathbf{R}^N)\}$ est alors un drap brownien et pour toute fonction $f \in L^2(\mathbf{R}^N)$, l'intégrale $\int_{[0,t]} f(s).\mathbb{W}(ds)$ coincide avec l'intégrale d'Itô $\int_{[0,t]} f(s).dB_s$.

### Représentation à moyenne mobile

Mandelbrot et Van Ness ont défini le mouvement brownien fractionnaire comme le processus $B^H$ tel que pour tout $t \in \mathbf{R}_+$,

$$B_t^H = \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right].\mathbb{W}(ds) + \int_0^t (t-s)^{H-\frac{1}{2}}.\mathbb{W}(ds)$$

où $H \in ]0,1[$.
Cette intégrale est une *représentation à moyenne mobile* du fBm. Elle n'est pas unique. Par exemple, on peut voir aisément que

$$B_t^H = \int_{\mathbf{R}} \left[|t-s|^{H-\frac{1}{2}} - |s|^{H-\frac{1}{2}}\right].\mathbb{W}(ds)$$

en est une autre.

### Représentation harmonisable

Tout d'abord, on définit la transformée de Fourier du processus isonormal complexe $\{W(h);\ h \in L^2_{\mathbf{C}}(\mathbf{R}^N)\}$, comme le processus gaussien centré $\hat{W} = \left\{\hat{W}(h);\ h \in L^2_{\mathbf{C}}(\mathbf{R}^N)\right\}$ tel que

$$\forall h \in L^2_{\mathbf{C}}(\mathbf{R}^N); \quad \hat{W}(h) = W(\hat{h})$$

où $\hat{h}$ désigne la transformée de Fourier de la fonction $h$.

Il est alors bien connu que pour $H \in ]0,1[$, le processus $X = \{X_t;\ t \in \mathbf{R}_+\}$ défini par

$$X_t = \int_{\mathbf{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+\frac{1}{2}}}.\hat{\mathbb{W}}(d\xi)$$

est un mouvement brownien fractionnaire (à une constante multiplicative près).

## 1.2 Propriétés

Dans le cas $H \neq \frac{1}{2}$, de nombreuses propriétés du mouvement brownien sont perdues. La corrélation entre les accroissements fait que le mouvement brownien fractionnaire n'est pas une semi-martingale. C'est pourquoi l'intégration stochastique par rapport au fBm n'entre pas dans le cadre habituel de l'intégration stochastique par rapport aux semi-martingales.

Cependant, subsistent deux propriétés fondamentales qui font du brownien fractionnaire le processus fractal le plus important :

- l'auto-similarité

$$\forall a \in \mathbf{R}_+; \quad \left\{B^H_{at}\right\}_{t\in\mathbf{R}_+} \stackrel{(d)}{=} a^H \left\{B^H_t\right\}_{t\in\mathbf{R}_+}$$

- la stationnarité des accroissements

$$\forall h \in \mathbf{R}_+; \quad \left\{B^H_{t+h} - B^H_h\right\}_{t\in\mathbf{R}_+} \stackrel{(d)}{=} \left\{B^H_t - B^H_0\right\}_{t\in\mathbf{R}_+}$$

Inversement, le mouvement brownien fractionnaire est l'unique processus gaussien centré continu auto-similaire et à accroissements indépendants.

## 1.3 Régularité höldérienne

Pour mesurer la régularité des trajectoires d'un processus, il est désormais classique d'utiliser les deux exposants suivants (cf [2], [3]) :

- l'*exposant de Hölder ponctuel*

$$\boldsymbol{\alpha}_X(t_0) = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|X_t - X_s|}{\rho^\alpha} < \infty\right\}$$

- l'*exposant de Hölder local*

$$\tilde{\boldsymbol{\alpha}}_X(t_0) = \sup\left\{\alpha : \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|X_t - X_s|}{|t-s|^\alpha} < \infty\right\}$$

Ces deux exposants sont complémentaires l'un de l'autre. De manière générale, on a $\tilde{\boldsymbol{\alpha}} \leq \boldsymbol{\alpha}$. Mais ces deux exposants peuvent être différents comme dans l'exemple déterministe de la fonction $t \mapsto |t|^\gamma \sin\frac{1}{|t|^\delta}$, où $\tilde{\alpha}(0) = \frac{\gamma}{1+\delta} < \alpha(0) = \gamma$.

Les exposants de Hölder ponctuel et local en tout point de la trajectoire du mouvement brownien fractionnaire sont presque surement égaux à $H$, i.e. quel que soit $t_0 \in \mathbf{R}_+$, on a presque surement $\tilde{\boldsymbol{\alpha}}_{B^H}(t_0) = \boldsymbol{\alpha}_{B^H}(t_0) = H$.
Dans l'article [12] composant le chapitre IV, on montre qu'on peut intervertir les quantificateurs "quel que soit" et "presque surement". Ainsi, on a presque surement

$$\forall t_0 \in \mathbf{R}_+; \quad \tilde{\boldsymbol{\alpha}}_{B^H}(t_0) = \boldsymbol{\alpha}_{B^H}(t_0) = H$$

# 2 Le mouvement brownien multifractionnaire

Dans l'optique des applications, la régularité constante le long des trajectoires du mouvement brownien fractionnaire peut être vu comme une limitation importante.

Le *mouvement brownien multifractionnaire* (mBm) a été introduit indépendamment dans [5] et [22]. Il est défini comme une généralisation du mouvement brownien fractionnaire dans lequel le paramètre d'auto-similarité $H \in ]0,1[$ est remplacé par une application $t \mapsto H(t)$ à valeurs dans $]0,1[$.

## 2.1 Les différentes représentations

Les deux groupes d'auteurs ont défini leur processus sous la forme de représentations différentes, et ce n'est que plus tard, dans [7], que fut montré que ces deux définitions sont deux représentations d'un même processus.

### Représentation à moyenne mobile

Peltier et Levy-Vehel ([22]) ont défini le mouvement brownien multifractionnaire $X = \{X_t;\ t \in \mathbf{R}_+\}$ à partir de la représentation à moyenne mobile du brownien fractionnaire

$$X_t = \int_{-\infty}^{0} \left[(t-s)^{H(t)-\frac{1}{2}} - (-s)^{H(t)-\frac{1}{2}}\right].\mathbb{W}(ds) + \int_0^t (t-s)^{H(t)-\frac{1}{2}}.\mathbb{W}(ds)$$

où $H : \mathbf{R}_+ \mapsto ]0,1[$ est une fonction mesurable.
De même que dans le cas du fBm, on peut voir que

$$X_t = \int_{\mathbf{R}} \left[|t-s|^{H(t)-\frac{1}{2}} - |s|^{H(t)-\frac{1}{2}}\right].\mathbb{W}(ds)$$

est une autre représentation à moyenne mobile du mouvement brownien multifractionnaire.

Pour définir le mBm, il n'est pas nécessaire de fixer d'hypothèses sur la fonction $H$. Cependant, pour étudier les propriétés de ce processus, ne serait-ce que l'existence d'une modification continue, on supposera que $H$ est $\beta$-hölderienne (avec $\beta > 0$).

### Représentation harmonisable

Benassi, Jaffard et Roux ([5]) ont défini le mouvement brownien multifractionnaire à partir de la représentation harmonisable du brownien fractionnaire, c'est-à-dire par

$$X_t = \int_{\mathbf{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+\frac{1}{2}}}.\hat{\mathbb{W}}(d\xi) \tag{I.2}$$

Ce processus est, à un facteur déterministe multiplicatif près, indistingable du processus défini par Peltier et Levy-Vehel.

### Covariance du mBm

La covariance du mouvement brownien multifractionnaire, sous la forme (I.2), a été calculée dans [4].

$$\forall s,t \in \mathbf{R}_+; \quad E\left[X_s X_t\right] = D\left[H(s) + H(t)\right]\left[s^{H(s)+H(t)} + t^{H(s)+H(t)} - |t-s|^{H(s)+H(t)}\right]$$

où $D : x \mapsto \int_{\mathbf{R}_+} \frac{1-e^{iu}}{|u|^{x+1}}.du$.

La connaissance de cette fonction de covariance, compte tenu du fait que la fonction multiplicative est lisse, permet de dégager des propriétés intéressantes du mBm.

## 2.2 Auto-similarité asymptotique locale

La substitution du paramètre constant $H$ dans la définition du mouvement brownien fractionnaire, par une fonction $H : \mathbf{R}_+ \mapsto ]0,1[$ fait perdre les deux propriétés caractéristiques que sont l'auto-similarité et la stationnarité des accroissements. Cependant, on garde une forme plus faible des ces deux propriétés qui est l'*auto-similarité asymptotique locale.*

Dans le cas où $H(t_0) < \beta$, la loi du processus $\left\{\frac{X_{t_0+\rho.u}-X_{t_0}}{\rho^{H(t_0)}};\ u \in \mathbf{R}_+\right\}$ converge au sens faible vers la loi d'un mouvement brownien fractionnaire de paramètre $H(t_0)$, lorsque $\rho$ tend vers 0.
Cette propriété est étudiée plus en détail au chapitre II dans le cadre multi-paramétré. On y montre que l'hypothèse $H(t_0) < \beta$ peut être remplacée par $H(t_0) < \tilde{\beta}(t_0)$, où $\tilde{\beta}(t_0) = \tilde{\alpha}_H(t_0)$ est l'exposant de Hölder local de la fonction $H$ au point $t_0$. Par ailleurs, lorsque $\tilde{\beta}(t_0) < H(t_0)$, la loi limite peut être différente de celle d'un fBm.

## 2.3 Régularité höldérienne

La définition du mouvement brownien multifractionnaire a été motivé par l'intérêt applicatif d'une régularité locale variant le long des trajectoires du processus.
La substitution de la constante $H$ dans la definition du fBm, par une fonction, ne conduit pas de manière évidente à une régularité non constante.

Sous l'hypothèse $H(t_0) < \beta$, les exposants de Hölder ponctuel et local en $t_0 \in \mathbf{R}_+$ vérifient

$$\boldsymbol{\alpha}_X(t_0) = \tilde{\boldsymbol{\alpha}}_X(t_0) = H(t_0) \quad \text{p.s.}$$

Plusieurs améliorations ont été obtenues sur ce résultat :

- Dans l'article [11] composant le chapitre II, l'hypothèse $H(t_0) < \beta$ a été supprimée. Plus précisément, pour tout $t_0 \in \mathbf{R}_+$,

$$\left\{ \begin{array}{ll} \boldsymbol{\alpha}_X(t_0) = \beta(t_0) \wedge H(t_0) & \text{p.s.} \\ \tilde{\boldsymbol{\alpha}}_X(t_0) = \tilde{\beta}(t_0) \wedge H(t_0) & \text{p.s.} \end{array} \right. \tag{I.3}$$

  où $\beta(t_0)$ et $\tilde{\beta}(t_0)$ sont les exposants de Hölder ponctuel et local de la fonction $H$ en $t_0$. On constate alors que les irrégularités de la fonction déterministe $H$ peuvent être héritées par la trajectoire du mBm.

- D'autre part, dans l'article [12] composant le chapitre IV, si la fonction $t \to \tilde{\beta}(t)$, où $\tilde{\beta}(t)$ désigne l'exposant de Hölder local de $H$ en $t$, est continue, on montre que presque surement

$$\forall t_0 \in \mathbf{R}_+; \quad \tilde{\boldsymbol{\alpha}}_X(t_0) = H(t_0) \wedge \tilde{\beta}(t_0)$$

  de plus, sous l'hypothèse $\forall t_0 \in \mathbf{R}_+, H(t_0) < \beta$, on montre que presque surement

$$\forall t_0 \in \mathbf{R}_+; \quad \boldsymbol{\alpha}_X(t_0) = \tilde{\boldsymbol{\alpha}}_X(t_0) = H(t_0)$$

  Par rapport aux premiers résultats concernant le mBm, on peut alors intervertir les quantificateurs "presque surement" et "quel que soit". Ce point difficile est largement traité dans le chapitre IV.

# 3 Les extensions à $N$ paramètres

Dans les années 1970, de nombreux travaux ont été entrepris dans le but de construire une *théorie générale des processus à deux indices*. En particulier, une théorie des martingales a été élaborée, ainsi qu'un calcul stochastique pour cette nouvelle classe de processus. Parmi les auteurs qui ont participé à ce chantier, citons D. Backry, R. Cairoli, X. Guyon, E. Merzbach, P. A. Meyer, D. Nualart, B. Prum, J. B. Walsh, E. Wong, M. Zakai, et d'autres, la liste n'est pas exhaustive. L'ouvrage récent de D. Khoshnevisan ([19]) constitue un précieux recueil des principaux résultats établis depuis cette période jusqu'à nos jours.

A la lumière de cette théorie, il paraît intéressant de définir des extensions à plusieurs paramètres du mouvement brownien (multi-)fractionnaire.

Le mouvement brownien possède deux extensions classiques pour des paramètres variant dans $\mathbf{R}_+^N$

- le *mouvement brownien de Lévy* est le processus gaussien centré de covariance

$$E\left[X_s X_t\right] = \frac{1}{2}\left[\|s\| + \|t\| - \|t-s\|\right]$$

  Il conduit à étendre la covariance incrémentale du mouvement brownien standard, $E\left[B_t - B_s\right]^2 = |t-s|$ pour $s, t \in \mathbf{R}_+$, en $E\left[X_t - X_s\right]^2 = \|t-s\|$ pour $s, t \in \mathbf{R}_+^N$.

- le *drap brownien* est le processus gaussien centré de covariance

$$E\left[X_s X_t\right] = \prod_{i=1}^{N} s_i \wedge t_i$$

  qui étend de manière tensorielle la covariance du brownien.

## 3.1 Les extensions du mouvement brownien fractionnaire

De la même manière, le mouvement brownien fractionnaire possède deux extensions classiques à plusieurs paramètres

- le *mouvement brownien fractionnaire de Lévy* est le processus gaussien centré de covariance

$$E\left[X_s X_t\right] = \frac{1}{2}\left[\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right]$$

  où $H \in ]0,1[$.
  Il est considéré par Adler dans [1] et par Kahane dans [15].

- le *drap brownien fractionnaire*, introduit par Kamont dans [16], est le processus gaussien centré de covariance

$$E\left[X_s X_t\right] = \prod_{i=1}^{N} \frac{1}{2}\left[s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}\right]$$

  où $(H_1, \ldots, H_N) \in ]0,1[^N$.

Ces deux extensions sont étudiées en détail au chapitre II. Comme le fBm standard, ces processus possèdent une représentation à moyenne mobile et une représentation harmonisable. Celles-ci constituent la base des deux extensions du mouvement brownien multifractionnaire.

Le mouvement fractionnaire de Lévy et le drap brownien fractionnaire héritent tous les deux des propriétés caractéristiques du fBm standard.

### Auto-similarité

Les deux extensions à $N$ paramètres du fBm, sont auto-similaires

- le mouvement fractionnaire de Lévy est auto-similaire d'indice $H$

$$\forall a \in \mathbf{R}_+; \quad \left\{X_{a.t};\ t \in \mathbf{R}_+^N\right\} \stackrel{(d)}{=} \left\{a^H X_t;\ t \in \mathbf{R}_+^N\right\}$$

- le drap brownien fractionnaire est auto-similaire d'indice $\sum_i H_i$

$$\forall a \in \mathbf{R}_+; \quad \left\{X_{a.t};\ t \in \mathbf{R}_+^N\right\} \stackrel{(d)}{=} \left\{a^{\sum_i H_i} X_t;\ t \in \mathbf{R}_+^N\right\}$$

### Stationnarité des accroissements

La notion d'accroissements d'un processus dont les indices varient dans $\mathbf{R}_+^N$, est plus complexe que dans le cas d'un seul paramètre.
En effet, plutôt que définir l'accroissement du processus $X$ entre $s$ et $t$ dans $\mathbf{R}_+^N$ par la quantité $X_t - X_s$, on préfère

$$\Delta X_{s,t} = \sum_{r \in \{0,1\}^{\#I}} (-1)^{\#I - \sum_l r_l} X_{[s_i + r_i(t_i - s_i)]_{i \in I}} \tag{I.4}$$

où $I = \{i = 1, \ldots, N;\ s_i \neq t_i\}$.
Cette définition est explicitée dans le chapitre II, ainsi que dans le chapitre III dans le cadre des processus indexés par des ensembles.

Compte tenu de cette définition, le mouvement fractionnaire de Lévy et le drap brownien fractionnaire sont à accroissements stationnaires

$$\forall h \in \mathbf{R}_+^N; \quad \left\{\Delta X_{h,h+t};\ t \in \mathbf{R}_+^N\right\} \stackrel{(d)}{=} \left\{\Delta X_{0,t};\ t \in \mathbf{R}_+^N\right\}$$

### Régularité höldérienne

De même que pour les fonctions d'une variable réelle, pour étudier la régularité des fonctions à variable dans $\mathbf{R}_+^N$, on considère les exposants de Hölder ponctuel et local

$$\boldsymbol{\alpha}_X(t_0) = \sup\left\{\alpha :\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\rho^\alpha} < \infty\right\}$$

et

$$\tilde{\boldsymbol{\alpha}}_X(t_0) = \sup\left\{\alpha :\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t - s\|^\alpha} < \infty\right\}$$

Dans [11] composant le chapitre II, on montre qu'en tout point $t$ de la trajectoire du

- mouvement brownien fractionnaire de Lévy,

$$\boldsymbol{\alpha}_X(t) = \tilde{\boldsymbol{\alpha}}_X(t) = H \quad \text{p.s.}$$

- drap brownien fractionnaire,

$$\boldsymbol{\alpha}_X(t) = \tilde{\boldsymbol{\alpha}}_X(t) = \min_i H_i \quad \text{p.s.}$$

Comme dans le cas du fBm standard, la régularité est constante le long de la trajectoire de ces processus.

## 3.2 Les extensions du mouvement brownien multifractionnaire

Les deux extensions à $N$ paramètres du mouvement brownien fractionnaire, permettent de définir deux extensions du mouvement brownien multifractionnaire, l'une est isotrope et l'autre est un drap.

### Le champ brownien multifractionnaire

Dans la représentation à moyenne mobile du mouvement brownien fractionnaire de Lévy, on remplace la constante $H \in ]0,1[$ par une fonction $t \mapsto H(t)$ à valeurs dans $]0,1[$.
On définit le *champ brownien multifractionnaire* comme le processus $\{X_t;\ t \in \mathbf{R}_+^N\}$ tel que

$$X_t = \int_{\mathbf{R}^N} \left[ \|t-u\|^{H(t)-\frac{N}{2}} - \|u\|^{H(t)-\frac{N}{2}} \right] .\mathbb{W}(du)$$

où $\mathbb{W}$ est le bruit blanc de $\mathbf{R}^N$.
Dans [11], on montre que ce processus est, à une fonction déterministe multiplicative près, indistingable du processus $\{X_t;\ t \in \mathbf{R}_+^N\}$ tel que

$$X_t = \int_{\mathbf{R}^N} \frac{e^{i<t,\xi>} - 1}{\|\xi\|^{H(t)+\frac{N}{2}}} .\hat{\mathbb{W}}(d\xi) \tag{I.5}$$

où $\hat{\mathbb{W}}$ est la transformée de Fourier du bruit blanc de $\mathbf{R}^N$.
Ce processus avait déjà été considéré dans [5], mais uniquement sous sa forme harmonisable (I.5).

### Le drap brownien multifractionnaire

De même que dans le cas isotrope, partant de la représentation à moyenne mobile du drap brownien fractionnaire, on remplace la constante $H \in ]0,1[^N$ par une fonction $t \mapsto H(t)$ à valeurs dans $]0,1[^N$.
Le *drap brownien multifractionnaire* est défini comme le processus $\{X_t;\ t \in \mathbf{R}_+^N\}$ tel que

$$X_t = \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left[ |t_i - u_i|^{H_i(t)-\frac{1}{2}} - |u_i|^{H_i(t)-\frac{1}{2}} \right] .\mathbb{W}(du)$$

Dans [11], on montre que ce processus est, à une fonction déterministe multiplicative près, indistingable du processus $\{X_t;\ t \in \mathbf{R}_+^N\}$ tel que

$$X_t = \int_{\mathbf{R}^N} \prod_{m=1}^{N} \frac{e^{it_m \xi_m} - 1}{|\xi_m|^{H_m(t)+\frac{1}{2}}} .\hat{\mathbb{W}}(d\xi)$$

### Covariance des extensions du mBm

Pour étudier les propriétés de ces processus, il est important d'évaluer leur covariance. Dans [11], on montre qu'il existe des fonctions déterministes lisses $D_N^f$ et $D_N^s$, telles que

- la fonction de covariance du champ brownien multifractionnaire peut s'écrire

$$E[X_s X_t] = D_N^f(H(s)+H(t)) \left[ \|s\|^{H(s)+H(t)} + \|t\|^{H(s)+H(t)} - \|t-s\|^{H(s)+H(t)} \right]$$

- et la fonction de covariance du drap brownien fractionnaire

$$E[X_s X_t] = D_N^s (H(s)+H(t)) \prod_{m=1}^{N} \left[ s_m^{H_m(s)+H_m(t)} + t_m^{H_m(s)+H_m(t)} - |t_m - s_m|^{H_m(s)+H_m(t)} \right]$$

Le caractère lisse des fonctions $D_N^f$ et $D_N^s$ permet d'obtenir un comportement asymptotique de la covariance incrémentale $E[X_t - X_s]^2$ lorsque $s$ et $t$ sont proches. Plus précisémment, dans le cas du champ brownien multifractionnaire, il existe une fonction continue $\phi$ telle que

$$E[X_t - X_s]^2 \sim D_N^f (2H(t)) .\|t-s\|^{2H(t)} + \phi(t).|H(t) - H(s)|^2 \tag{I.6}$$

lorsque $\|t-s\| \to 0$.
Sous l'hypothèse $\sup_t H(t) < \beta$, considérée jusqu'alors dans les définitions du mBm, le premier terme est dominant dans cet équivalent asymptotique. Cependant, la présence du deuxième terme permet d'étudier la trajectoire du processus dans le cas où la fonction $H$ est irrégulière.
Ainsi, lorsque la valeur de la fonction $H$ en un point est inférieure à sa régularité, de l'équivalence

$$E[X_{t_n} - X_{s_n}]^2 \sim \phi(t).|H(t_n) - H(s_n)|^2$$

pour certaines suites $(s_n)_{n\in\mathbf{N}}$ et $(t_n)_{n\in\mathbf{N}}$ tendant vers $t$, on déduit que les propriétés du champ brownien multifractionnaire sont fonction de la régularité de $H$ et non pas de sa valeur (cf [11]).

### Auto-similarité asymptotique locale

Comme dans le cas d'indices dans $\mathbf{R}_+$, la substitution du paramètre constant $H$ par une fonction $t \mapsto H(t)$, fait perdre les deux propriétés d'auto-similarité et de stationnarité des accroissements. Dans [11], une propriété d'auto-similarité asymptotique locale est montré pour le champ et le drap brownien multifractionnaire.
Pour ces deux processus, on montre dans [11], sous certaines hypothèses, la convergence du processus $\left\{ \frac{\Delta X_{t_0, t_0+\rho.u}}{\rho^\alpha};\ u \in \mathbf{R}_+^N \right\}$ lorsque $\rho$ tend vers 0.

Plus précisemment, si $X$ est un champ brownien multifractionnaire, la loi du processus $Y^\alpha(\rho) = \left\{ Y_u^\alpha = \frac{X_{t_0+\rho.u} - X_{t_0}}{\rho^\alpha};\ u \in \mathbf{R}_+^N \right\}$ converge au sens faible si l'une des deux conditions suivantes est remplie :

1. $\begin{cases} \alpha = H(t_0) \\ H(t_0) < \inf_{u,v} \beta_{uv}(t_0) \end{cases}$
   où $\beta_{uv}(t_0) = \sup \left\{ \alpha :\ \lim_{\rho\to 0} \frac{|H(t_0+\rho.u) - H(t_0+\rho.v)|}{\rho^\alpha} = 0 \right\}$.
   La loi limite est celle d'un mouvement de Lévy fractionnaire $B^{H(t_0)}$.

2. $\begin{cases} \alpha = \inf_{u,v} \beta_{uv}(t_0) \\ H(t_0) > \inf_{u,v} \beta_{uv}(t_0) \\ \forall u, v; \quad \lim_{\rho\to 0} \frac{|H(t_0+\rho.u) - H(t_0+\rho.v)|}{\rho^{\inf_{u,v}\beta_{uv}(t_0)}} = \Gamma(u,v) \end{cases}$
   avec $(u,v) \mapsto \frac{\Gamma(u,v)}{\|u-v\|^{2\beta}}$ bornée sur $[a,b]^2$ pour un $\beta > 0$.
   La loi limite est la loi d'un processus gaussien $Y$ tel que

$$E[Y_u - Y_v]^2 = K.\Gamma(u,v)$$

Dans le cas $N = 1$, la première condition correspond à la propriété LASS, déjà connue du mBm ([5], [22]). Cependant, la deuxième condition montre une propriété différente du mBm dans le cas où la fonction $H$ est irrégulière. Dans ce cas, la grandeur pertinente est la régularité de $H$, et non pas sa valeur elle-même. Ainsi, la loi limite peut-être celle d'un processus dont les accroissements ne sont pas stationnaires ([11]). Ce cas n'entre donc pas dans le cadre général des processus tangents étudiés par K. Falconer dans [10].

Si $X$ est un drap brownien multifractionnaire, la loi du processus

$$Y^\alpha(\rho) = \left\{ Y_u^\alpha = \frac{\Delta X_{t_0,t_0+\rho.u}}{\rho^{\sum_i \alpha_i}};\ u \in \mathbf{R}_+^N \right\}$$

converge au sens faible si pour tout $i \in \{1, \dots, N\}$, l'une des deux conditions précédentes est vérifiée.

**Régularité höldérienne**

Dans les articles [11] et [12], composant les chapitres II et IV, on montre que la régularité des processus gaussiens peut être connue grâce à l'étude de la quantité $E\left[X_t - X_s\right]^2$ lorsque $s$ et $t$ sont proches.
Ainsi, le comportement asymptotique (I.6) permet d'évaluer la régularité de la trajectoire des processus. Selon la valeur de la fonction $H$ par rapport à sa régularité locale $\tilde{\beta}$, le premier terme de (I.6) domine le second, ou vice versa.

Dans [11], est évaluée en chaque point $t_0 \in \mathbf{R}_+^N$, la valeur presque sure des exposants de Hölder ponctuel et local

- du champ brownien multifractionnaire,

$$\begin{aligned} \boldsymbol{\alpha}_X(t_0) &= \beta(t_0) \wedge H(t_0) \quad \text{p.s.} \\ \tilde{\boldsymbol{\alpha}}_X(t_0) &= \tilde{\beta}(t_0) \wedge H(t_0) \quad \text{p.s.} \end{aligned}$$

- et du drap brownien multifractionnaire,

$$\begin{aligned} \boldsymbol{\alpha}_X(t_0) &= \beta(t_0) \wedge \min_i H_i(t_0) \quad \text{p.s.} \\ \tilde{\boldsymbol{\alpha}}_X(t_0) &= \tilde{\beta}(t_0) \wedge \min_i H_i(t_0) \quad \text{p.s.} \end{aligned}$$

où $\beta(t_0)$ et $\tilde{\beta}(t_0)$ sont les exposants de Hölder ponctuel et local de la fonction $H$ en $t_0$.

On constate que dans le cas d'une fonction $H$ irrégulière, la régularité de la trajectoire des deux processus est égale à celle de $H$. Ainsi, ces processus multifractionnaires peuvent être vus comme une passerelle entre le monde des fonctions déterministes et celui des processus stochastiques. En particulier, les travaux sur l'existence de fonctions à régularité locale prescrite ([8]) peuvent être utilisés pour obtenir un processus à régularité locale prescrite.

## 4 Les processus indexés par des ensembles

Dans la continuité des travaux sur les processus à deux indices, la motivation pour indexer des processus stochastiques par des ensembles, est très naturelle. En effet, chaque processus $X = \{X_t;\ t \in \mathbf{R}_+^N\}$, peut déjà être considéré comme indexé par les rectangles $[0, t]$ de $\mathbf{R}_+^N$.

Par exemple, dans le cas bien connu du drap brownien, processus gaussien centré indexé par $\mathbf{R}_+^N$ dont la covariance est

$$\forall s,t \in \mathbf{R}_+^N; \quad E\left[B_s B_t\right] = s \curlywedge t$$

où $s \curlywedge t$ désigne l'élément $(s_i \wedge t_i)_i$ de $\mathbf{R}_+^N$, cette identification permet l'écriture plus ramassée de la covariance du processus

$$\forall s,t \in \mathbf{R}_+^N; \quad E\left[B_{[0,s]} B_{[0,t]}\right] = Leb\left([0,s] \cap [0,t]\right)$$

où *Leb* désigne la mesure de Lebesgue de $\mathbf{R}_+^N$. Cette expression est en fait la covariance du *mouvement brownien indexé par des ensembles* (ou bruit blanc de $\mathbf{R}_+^N$), dans les cas particulier des rectangles de $\mathbf{R}_+^N$ (cf paragraphe 1.1, [1] et [19]).

D'autre part, le cadre des processus indexés par des ensembles semble très intéressant dans l'optique des applications en modélisation stochastique. En effet, pour diverses raisons, les données réelles sont souvent incomplètes (*données censurées*). C'est le cas par exemple lors de l'étude statistique de l'influence d'un certain facteur sur la longévité d'une population, ou de la modélisation aléatoire d'un terrain vu par le radar d'un avion. Dans de tels cas, la modélisation basée simplement sur des processus à plusieurs paramètres s'avère difficile à mettre en oeuvre, alors que le cadre *set-indexed* paraît tout indiqué.

Une théorie générale des processus indexés par des ensembles est en plein développement. La theorie des martingales a été élaborée ces dernières années ([14]), ainsi qu'un calcul stochastique ([26]).

La construction d'une extension *set-indexed* du mouvement brownien fractionnaire paraît intéressante pour les applications à double titre : ce processus est alors le plongement du fBm, bien connu pour son intérêt en modélisation, dans le cadre prometteur des processus indexés par des ensembles.

## 4.1 Collections d'indices, accroissements

La famille d'ensembles $\mathcal{A}$ indexant les processus, parties d'un espace de mesure $\mathcal{T}$, est choisie vérifiant un certain nombre de propriétés qu'on n'expose pas dans cette partie (cf [14], [13]). Les exemples typiques de telles collections sont :

- les rectangles $[0,t]$ de $\mathbf{R}_+^N$,
- les ensembles minimaux (*lower sets*),
  i.e. tels que pour tout $A \in \mathcal{A}$, $(t \in A \Rightarrow [0,t] \subset A)$.

Pour tout processus stochastique $X = \{X_U;\ U \in \mathcal{A}\}$, on définit

- la famille $\mathcal{C}$ de parties de $\mathcal{T}$, de la forme $C = U \setminus \left(\bigcup_{1\leq i\leq n} U_i\right)$, où $U, U_1, \ldots, U_n \in \mathcal{A}$,
- et le processus accroissement de $X$, $\Delta X = \{\Delta X_C;\ C \in \mathcal{C}\}$ par la formule d'inclusion-exclusion

$$\Delta X_C = X_U - \sum_{i=1}^{n} X_{U \bigcap U_i} + \sum_{i<j} X_{U \bigcap (U_i \bigcap U_j)} - \cdots + (-1)^n X_{U \bigcap \left(\bigcap_{1\leq i\leq n} U_i\right)} \qquad \text{(I.7)}$$

Dans le cas particulier où $\mathcal{A}$ est l'ensemble des rectangles de $\mathbf{R}_+^N$, la définition des accroissements (I.7) coincide avec l'expression (I.4) des accroissements d'un processus à $N$ paramètres.

La sous-famille $\mathcal{C}_0$ de $\mathcal{C}$, constituée des ensembles de la forme $C = U \setminus V$, où $U, V \in \mathcal{A}$, joue un rôle important dans la suite.

## 4.2 Définition d'un mouvement brownien fractionnaire indexé par des ensembles

Rappelons que le mouvement brownien fractionnaire indexé par des réels, est défini comme le processus gaussien centré tel que

$$\forall s, t \in \mathbf{R}_+; \quad E\left[B_t^H - B_s^H\right]^2 = |t-s|^{2H}$$

De même que les accroissements $t-s$ de $\mathbf{R}_+$, sont remplacés par les éléments de $\mathcal{C}$, l'idée naturelle pour définir une extension *set-indexed* du fBm est d'exiger la propriété

$$\forall C \in \mathcal{C}; \quad E\left[\Delta X_C\right]^2 = m(C)^{2H}$$

Cependant, dans l'article [13] composant le chapitre III, on montre que le seul processus gaussien vérifiant une telle propriété, est le mouvement brownien, pour lequel $H = \frac{1}{2}$ (théorème 4.3).
On choisit alors de définir le processus gaussien centré $B^H$ tel que

$$\forall U, V \in \mathcal{A}; \quad E\left[B_U^H - B_V^H\right]^2 = m(U \bigtriangleup V)^{2H}$$

Ce processus est appelé *mouvement brownien fractionnaire indexé par des ensembles (SifBm)*, de paramètre d'autosimilarité $H \in ]0,1[$.
On montre alors, dans le lemme 3.2 de [13], que pour le SifBm

$$\forall C \in \mathcal{C}_0; \quad E\left[\Delta X_C\right]^2 = m(C)^{2H}$$

L'existence du SifBm est prouvée dans [13]. Pour deux raisons différentes, sa définition peut être vue comme une extension du fBm classique, au cadre dans lequel les indices du processus sont des ensembles. La première est que dans le cas particulier où $H = \frac{1}{2}$, le processus $B^{\frac{1}{2}}$ est le *set-indexed Brownian motion* déjà étudié dans [1]. La deuxième raison est que dans le cas où la collection d'ensembles $\mathcal{A}$ est constituée des rectangles de $\mathbf{R}_+^N$, le processus $B^H$ n'est autre que le fBm standard pour $N = 1$. Cependant pour $N > 1$, le processus à $N$ paramètres ainsi défini n'est ni un mouvement fractionnaire de Lévy, ni un drap brownien fractionnaire. Pourtant, dans un certain sens, il possède les proriétés fractales qu'il est raisonnable de lui exiger.

## 4.3 Propriétés fractales du SifBm

Comme il a été rappelé dans la section 1, le mouvement brownien fractionnaire possède les deux propriétés fractales caractéristiques qui lui confèrent son rôle très particulier : il est auto-similaire et ses accroissements sont stationnaires.

### Stationnarité des accroissements

La notion d'accroissements stationnaires pour un processus indexé par des ensembles, peut être définie dans des sens plus ou moins forts. Plusieurs d'entre eux sont évoqués au chapitre III. Le SifBm ne possède cette propriété que sous la forme faible suivante.

Le SifBm possède la propriété de $\mathcal{C}_0$-*stationnarité (ou stationnarité de ses accroissements* $\mathcal{C}_0$*)*, dans le sens où pour tous $C$ et $C'$ dans $\mathcal{C}_0$ tels que $m(C) = m(C')$, on a $\Delta B_C^H \stackrel{(d)}{=} \Delta B_{C'}^H$.

**Auto-similarité**

Pour étudier l'auto-similarité d'un processus indexé par des ensembles, il est nécessaire de munir la collection d'indices $\mathcal{A}$, de l'opération d'un groupe $G$, de supposer qu'elle peut être étendue de telle sorte que

$$\forall U, V \in \mathcal{A}, \forall g \in G; \quad \begin{aligned} g.(U \cup V) &= g.U \cup g.V \\ g.(U \setminus V) &= g.U \setminus g.V \end{aligned}$$

et qu'il existe une fonction $\mu : G \to \mathbf{R}_+$ telle que

$$\forall U \in \mathcal{A}, \forall g \in G; \quad m(g.U) = \mu(g).m(U)$$

On montre dans [13] que le SifBm est auto-similaire de paramètre $H$, i.e. pour tout $g \in G$,

$$\left\{B^H_{g.U};\ U \in \mathcal{A}\right\} \stackrel{(d)}{=} \left\{\mu(g)^H.B^H_U;\ U \in \mathcal{A}\right\}$$

## 4.4 Pseudo-caractérisation du SifBm

De même que le mouvement brownien fractionnaire est l'unique processus gaussien centré indexé par $\mathbf{R}_+$, qui soit à la fois auto-similaire et à accroissements stationnaires, il est intéressant de considérer les processus indexés par des ensembles, qui possèdent les deux propriétés fractales précédentes.

Dans [13], on montre que si un processus indexé par des ensembles $X = \{X_U;\ U \in \mathcal{A}\}$ vérifie les deux propriétés :

1. auto-similarité de paramètre $H$
2. $\mathcal{C}_0$-stationnarité

alors la fonction de covariance entre deux ensembles $U$ et $V$ tels que $U \subset V$ est

$$E\left[X_U X_V\right] = K\ \left[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}\right]$$

Ce résultat montre que même si les deux propriétés fractales considérées ne caractérisent pas complètement la loi d'un processus *set-indexed*, elles imposent à sa covariance d'être égale à celle du SifBm au moins entre les ensembles inclus l'un dans l'autre.

## 4.5 Continuité

Pour des processus indexés par des ensembles, la notion de continuité est plus complexe que dans le cas où les indices sont dans $\mathbf{R}^N_+$. En effet, même le simple mouvement brownien, si la collection d'ensembles $\mathcal{A}$ est trop grande, ne possède pas de version continue sur $\mathcal{A}$. Ce fait est largement exposé dans [1].

Dans [13], on montre que le SifBm est continue sur les mêmes collections d'indices que le mouvement brownien indexé par des ensembles (théorème 5.1).

En particulier, le SifBm indexé par les rectangles $\left\{[0,t];\ t \in \mathbf{R}^N_+\right\}$ admet une modification continue.

### 4.6 SifBm sur les flots

La notion de *flot* est très importante pour réduire les problèmes de processus indexés par des ensembles, à des problèmes de processus indexés par $\mathbf{R}_+$. En particulier, la connaissance d'un processus sur tous les flots permet d'accéder à celle du processus en entier (cf [13], lemme 6.3).

On note $\tilde{\mathcal{A}}(u)$ l'ensemble des intersections finies d'unions finies d'éléments de $\mathcal{A}$. On désigne par flot, toute application croissante $f : [a,b] \to \tilde{\mathcal{A}}(u)$.

Dans [13], on montre que pour tout flot $f$, le processus $X^f = \left\{X_{f(s)};\ s \in [a,b]\right\}$, projection d'un SifBm $X$ sur $f$, est un mouvement brownien fractionnaire standard changé de temps (proposition 6.4).
Ce résultat est en fait valable pour tout processus auto-similaire dont les accroissements $\mathcal{C}_0$ sont stationnaires.

Dans les cas du mouvement brownien fractionnaire de Lévy et du drap brownien fractionnaire, seules les projections de ces processus sur certaines droites étaient des fbm.

À la lumière de ces résultats, il nous a paru naturel de définir une extension plus générale du mouvement brownien fractionnaire indexé par des ensembles. On appelle donc *general set-indexed fractional Brownian motion* (mouvement brownien fractionnaire indexé par des ensembles, général), un processus gaussien auto-similaire d'indice $H \in ]0,1[$, dont les accroissements $\mathcal{C}_0$ sont stationnaires.
Un tel processus existe, puisque le SifBm en est un. La projection d'un tel processus sur un flot est un fBm standard changé de temps. Et la covariance entre deux ensembles $U, V \in \mathcal{A}$ tels que $U \subset V$ est

$$E\left[X_U.X_V\right] = K\ \left[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}\right]$$

où $K$ est une constante positive.

## 5 Etude fine de la régularité de processus gaussiens

La notion de régularité est importante dans les problèmes nécessitant une interpolation. En traitement d'images, par exemple, les méthodes visant à reproduire sur les points interpolés, la régularité des données de départ, permet de réduire cette impression de flou couramment observée avec les techniques simplistes habituelles (cf super-resolution, ondelette, krigeage)

Ainsi, la régularité des trajectoires d'un processus stochastique apparaît comme un paramètre essentiel dans le cadre d'une modélisation aléatoire. Le mouvement brownien multifractionnaire, en tant que processus paramétré par la seule régularité de ses trajectoires, semble alors un processus très important en vue des applications.

Dans le cadre de la modélisation de certains phénomènes dont les fluctuations peuvent être considérée comme aléatoires, on peut être amené à considérer l'action d'opérateur intégro-différentiels sur le mouvement brownien multifractionnaire. C'est par exemple le cas lorsqu'on simule les variations des cours boursiers par une équation différentielle stochastique dirigé par un processus dérivé du mBm.

### 5.1 Des exposants de Hölder à l'analyse 2-microlocale

Comme rappelé dans la section 1.3, on peut considérer les exposants de Hölder ponctuel et local $\alpha_f(t_0)$ et $\tilde{\alpha}_f(t_0)$ de toute application continue $f$, en $t_0 \in \mathbf{R}$.
L'exposant ponctuel n'est pas stable sous l'action des opérateurs pseudo-différentiels.

Reprenons l'exemple du chirp

$$f: \begin{array}{l} \mathbf{R}_+ \to \mathbf{R} \\ x \mapsto x^\gamma \sin \frac{1}{x^\delta} \end{array}$$

où $\gamma > \delta + 1$ et $\delta > 0$.
On a $\alpha_f(0) = \gamma$.
D'autre part, au voisinage de 0, on a $f'(x) \sim x^{\gamma-\delta-1} \cos \frac{1}{x^\delta}$, et donc $\alpha_{f'}(0) = \gamma - \delta - 1$.

Répondre à la question :

— Comment prédire la valeur de $\alpha_{f'}$ à partir de $\alpha_f$ ?

est l'objet de l'analyse 2-microlocale, introduite par J.M. Bony dans le cadre de la théorie des EDP.

Selon la formulation de Kolwankar-Lévy Véhel-Seuret, pour tout $t_0$, et tous $s$, $s'$ tels que

$$\begin{cases} 0 < s + s' < 1 \\ s < 1 \\ s' < 0 \end{cases}$$

on définit la classe $C_{t_0}^{s,s'}$ comme l'ensemble des fonctions continues $\varphi$ telles que

$$\limsup_{\rho \to 0} \sup_{t,u \in B(t_0,\rho)} \frac{|\varphi(t) - \varphi(u)|}{\|t-u\|^{s+s'} \rho^{-s'}} < \infty$$

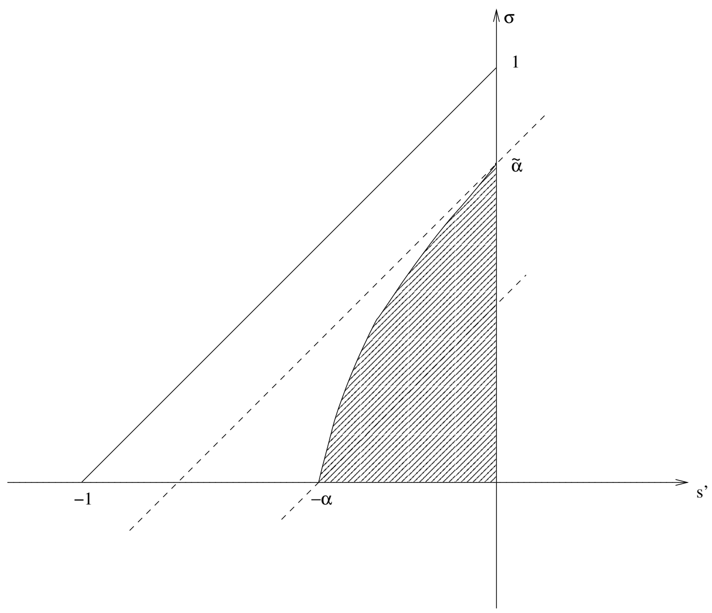
Dans la suite, on choisit de considérer les paramètres $(\sigma = s + s', s')$ plutôt que $(s, s')$. Dans le triangle

$$\begin{cases} 0 < \sigma < 1 \\ s' < 0 \\ \sigma - s' < 1 \end{cases} \tag{I.8}$$

on considère alors l'application

$$s' \mapsto \sigma_{t_0}(s') = \sup \left\{ \sigma : \limsup_{\rho \to 0} \sup_{t,u \in B(t_0,\rho)} \frac{|\varphi(t) - \varphi(u)|}{\|t-u\|^{\sigma} \rho^{-s'}} < \infty \right\}$$

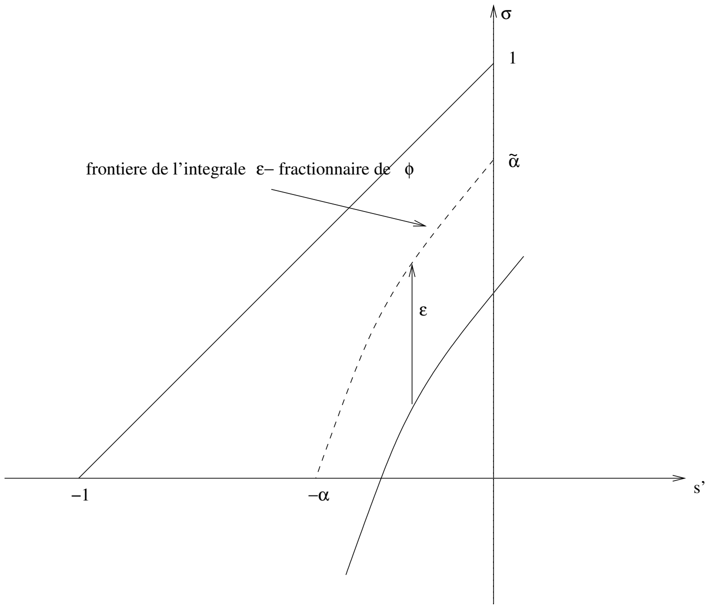
Cette application continue est la *frontière 2-microlocale* de $f$ en $t_0$.



Sur cette représentation de la frontière de $f$ en $t_0$, on visualise la propriété valable dans le triangle (I.8)

$$\forall s' < 0; \quad \tilde{\alpha}(t_0) \leq \sigma_{t_0}(s') - s' \leq \alpha(t_0)$$

La notion de frontière 2-microlocale permet de répondre à la question de la recherche de l'exposant ponctuel de la dérivée du chirp dans l'exemple introductif.
En effet, la frontière 2-microlocale des dérivées et intégrations fractionnaires d'une application $\varphi$, sont obtenues par translation de la frontière de $\varphi$ dans la direction de l'axe $\sigma$.




## 5.2 Frontière 2-microlocale des processus gaussiens

Le cadre de l'analyse 2-microlocale peut être utilisé pour étudier la régularité des trajectoires d'un processus stochastique.
Pour tout processus continu $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$, on définit alors la *frontière 2-microlocale* de $X$ en $t_0 \in \mathbf{R}_+^N$ comme l'application $s' \mapsto \boldsymbol{\sigma}_{t_0}(s')$ telle que pour tout $s' < 0$

$$\boldsymbol{\sigma}_{t_0}(s') = \sup\left\{\sigma;\ \limsup_{\rho\to 0}\ \sup_{s,t\in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^{\sigma}\rho^{-s'}} < \infty\right\}$$

De même que les exposants ponctuel et local $\boldsymbol{\alpha}(t_0)$ et $\tilde{\boldsymbol{\alpha}}(t_0)$, la frontière 2-microlocale est aléatoire. Dans l'article [12], on montre que sous certaines hypothèses, ces trois grandeurs possèdent des valeurs presque sures.

### Valeur presque sure de la frontière 2-microlocale en un point

On suppose que le processus $X$ est gaussien et centré. Dans ce cas, il est bien connu que l'étude des quantités $E\left[X_U - X_V\right]^2$ permet d'appréhender certains comportements presque surs des trajectoires (cf [9]).
On définit alors un nouveau type d'exposants déterministes :

- l'*exposant de Hölder local déterministe*

$$\tilde{\mathbb{\alpha}}(t_0) = \sup\left\{\alpha;\ \limsup_{\rho\to 0}\ \sup_{s,t\in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\alpha}} < \infty\right\}$$

- la *frontière 2-microlocale déterministe* $s' \mapsto \mathbb{\sigma}_{t_0}(s')$

$$\mathbb{\sigma}_{t_0}(s') = \sup \left\{ \sigma;\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\sigma} \rho^{-2s'}} < \infty \right\}$$

Dans [12], on montre que quel que soit $t_0 \in \mathbf{R}_+^N$, la frontière 2-microlocale en $t_0$ de la trajectoire de $X$ est égale presque surement au graphe de l'application $s' \mapsto \mathbb{\sigma}_{t_0}(s')$.

Le problème de l'interversion des quantificateurs "quel que soit" et "presque surement" a été investigué dans [12]. Il s'avère difficile à résoudre et les résultats dans cette voie demeurent incomplets.

### Résultats presque surs uniformément sur $\mathbf{R}_+^N$

On dispose d'un résultat uniforme sur la valeur presque sure de l'exposant local d'un processus gaussien centré $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$. Dans [12], on montre que presque surement

$$\forall t_0 \in \mathbf{R}_+^N; \quad \liminf_{u \to t_0} \tilde{\mathbb{\alpha}}(u) \leq \tilde{\boldsymbol{\alpha}}_X(t_0) \leq \limsup_{u \to t_0} \tilde{\mathbb{\alpha}}(u) \tag{I.9}$$

En conséquence, si l'application $t_0 \mapsto \tilde{\mathbb{\alpha}}(t_0)$ est continue, on a presque surement

$$\forall t_0 \in \mathbf{R}_+^N; \quad \tilde{\boldsymbol{\alpha}}_X(t_0) = \tilde{\mathbb{\alpha}}(t_0)$$

Concernant la frontière 2-microlocale, on ne connait qu'un encadrement uniforme de la valeur presque sure. Il manque, en effet, une minoration presque sure de $\boldsymbol{\sigma}_{t_0}(s')$ en terme de $\mathbb{\sigma}_{t_0}(s')$. On a presque surement

$$\forall t_0 \in \mathbf{R}_+^N, \forall s' < 0; \quad \tilde{\mathbb{\alpha}}(t_0) + s' \leq \boldsymbol{\sigma}_{t_0}(s') \leq \limsup_{u \to t_0} \mathbb{\sigma}_u(s') \tag{I.10}$$

## 5.3 Applications

Les résultats généraux sur la frontière 2-microlocale des processus gaussiens peuvent être appliqués aux processus classiques que sont le mouvement brownien fractionnaire ou la fonction de Weierstrass généralisée. De plus, ils permettent de connaître plus finement la régularité du brownien multifractionnaire. Enfin, dans [12], la frontière des solutions de certaines équations differentielles stochastiques est évaluée.

### Cas académique : le mouvement brownien fractionnaire

Comme le mouvement brownien fractionnaire est un processus continu, gaussien, centré tel que

$$\forall s, t \in \mathbf{R}_+; \quad E\left[X_t - X_s\right]^2 = |t-s|^{2H}$$

les résultats sur la frontière 2-microlocale peuvent être immédiatement appliqués. On a

$$\forall t_0 \in \mathbf{R}_+; \quad \tilde{\mathbb{\alpha}}(t_0) = H$$
$$\forall t_0 \in \mathbf{R}_+, \forall s' < 0; \quad \mathbb{\sigma}_{t_0}(s') = H + s'$$

Une application directe de (I.10) entraine, presque surement

$$\forall t_0 \in \mathbf{R}_+, \forall s' < 0; \quad \boldsymbol{\sigma}_{t_0}(s') = H + s'$$

En conséquence, on a presque surement

$$\forall t_0 \in \mathbf{R}_+; \quad \tilde{\boldsymbol{\alpha}}_{B^H}(t_0) = \boldsymbol{\alpha}_{B^H}(t_0) = H$$

#### Cas du mouvement brownien multifractionnaire

Le mouvement brownien multifractionnaire ne possède pas d'expression simple de la covariance incrémentale. Cependant, dans [11], on a montré un équivalent asymptotique de celle-ci, dans toute boule $B(t_0, \rho)$ lorsque $\rho$ tend vers 0

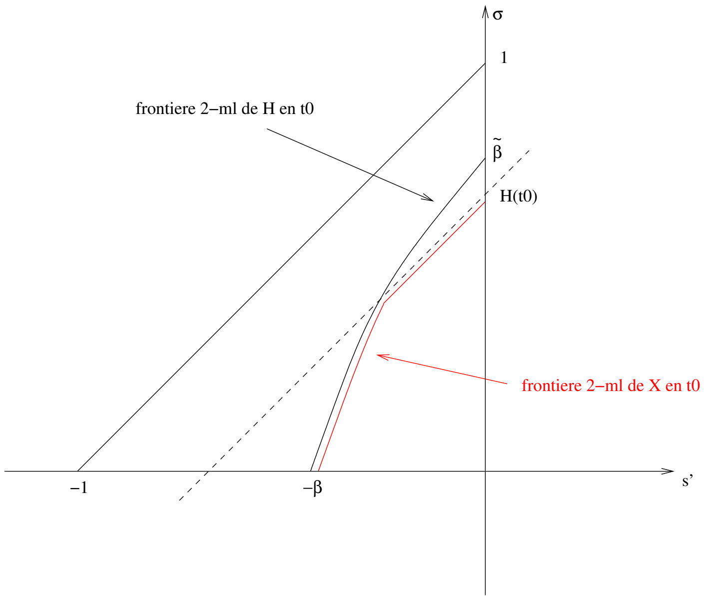
$$\forall s, t \in B(t_0, \rho); \quad E\left[X_t - X_s\right]^2 \sim K(t_0).|t-s|^{2H(t_0)} + L(t_0).\left[H(t) - H(s)\right]^2$$

Cet équivalent a permis de calculer en tout $t_0 \in \mathbf{R}_+$, les valeurs presque sures (I.3) des exposants de Hölder local et ponctuel. La question de l'interversion des quantificateurs "quel que soit" et "presque surement" était jusqu'alors sans réponse.

Dans [12], on évalue la frontière 2-microlocale presque sure du mBm en un point $t_0 \in \mathbf{R}_+$

$$\forall s' < 0; \quad \boldsymbol{\sigma}_{t_0}(s') = (H(t_0) + s') \wedge \beta_{t_0}(s')$$

où $s' \mapsto \beta_{t_0}(s')$ désigne la fonction 2-microlocale de $H$ en $t_0$.




En toute généralité, on ne connait pas de valeur presque sure de la frontière 2-microlocale du mBm, uniformément sur $\mathbf{R}_+$. Cependant, une application directe de (I.9) entraine, sous l'hypothèse de continuité de $t \mapsto \tilde{\beta}(t)$,

$$P\left\{\forall t_0 \in \mathbf{R}_+; \quad \tilde{\boldsymbol{\alpha}}(t_0) = H(t_0) \wedge \tilde{\beta}(t_0)\right\} = 1$$

De plus, dans le cas particulier où $\forall t_0 \in \mathbf{R}_+;\ H(t_0) < \tilde{\beta}(t_0)$, on a

$$P\left\{\forall t_0 \in \mathbf{R}_+, \forall s' < 0; \quad \boldsymbol{\sigma}_{t_0}(s') = H(t_0) + s'\right\} = 1$$

## 6 Quelles perspectives ?

### 6.1 Un mouvement brownien multifractionnaire indexé par des ensembles

Le mouvement brownien fractionnaire indexé par des ensembles est paramètré par l'unique nombre $H$, qui est également son paramètre d'auto-similarité. Des travaux commencés laissent penser que, dans un certain sens, $H$ mesure de plus la régularité des trajectoires du processus.

Définir un processus indexé par des ensembles, dont la régularité locale est fonction de l'ensemble indexant, constitue un objectif important au moins en vue des applications.

Pour cela, sur le modèle du cas multiparamétré, il est nécessaire d'obtenir des représentations du SifBm (intégrale ou autre) dans lesquel on espère pouvoir définir un *mouvement brownien multifractionnaire indexé par des ensembles*, $X = \{X_U;\ U \in \mathcal{A}\}$, en remplaçant la constante $H$ par $H(U)$, où $H$ est une certaine fonction régulière.

## 6.2 Estimation-simulation

Les travaux développés dans cette thèse ont mis en évidence un comportement irrégulier du mouvement brownien multifractionnaire (qu'il soit à plusieurs paramètres ou non). L'estimateur de la fonction $H$, construit dans l'hypothèse où $H$ est inférieure à sa régularité, n'est plus valable dans le cas général. En effet, estimer la régularité de la trajectoire du processus en $t$, ne fournit pas, dans le cas où la fonction $H$ est irrégulière, une estimation de la valeur de $H(t)$, mais uniquement de la régularité de $H$ en $t$. Il sera donc nécessaire de construire un estimateur de $H(t)$ dans ce cas.

Des algorithmes de simulation rapides sont, en dimension supérieure ($N > 1$), encore plus importants que pour des processus à indices réels. Les travaux d'Amar et Cragg sur la résolution rapide de systèmes de Toeplitz devraient permettre des gains très importants dans ce but.

## 6.3 Valeur presque sure uniforme de la frontière 2-microlocale du mBm

Dans [12], nous avons obtenu en tout point $t_0 \in \mathbf{R}_+^N$, la valeur presque sure de la frontière 2-microlocale en $t_0$ du mouvement brownien multifractionnaire. De plus, dans le cas où la fonction $H$ est régulière, cette frontière presque sure est uniforme sur $\mathbf{R}_+^N$ (on peut intervertir les quantificateurs "pour tout $t_0$" et "presque surement"). Cependant, un tel résultat uniforme n'a pas été obtenu dans le cas général. C'est en particulier, la minoration uniforme de la fonction 2-microlocale qui fait défaut.

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# Chapitre II

## From $N$-parameter fractional Brownian motions to $N$-parameter multifractional Brownian motions

Article à paraitre dans *Rocky Mountains Journal of Mathematics* 2005, auquel ont été rajoutés les deux paragraphes :

4.3 Directional Hölder exponents

4.4 Application of Dudley's theory

ainsi que le détail des preuves de la section 5.

# From $N$ parameter fractional Brownian motions to $N$ parameter multifractional Brownian motions

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**Abstract**

Multifractional Brownian motion is an extension of the well-known fractional Brownian motion where the Hölder regularity is allowed to vary along the paths. In this paper, two kind of multi-parameter extensions of mBm are studied: one is isotropic while the other is not. For each of these processes, a moving average representation, a harmonizable representation, and the covariance structure are given.
The Hölder regularity is then studied. In particular, the case of an irregular exponent function $H$ is investigated. In this situation, the almost sure pointwise and local Hölder exponents of the multi-parameter mBm are proved to be equal to the correspondent exponents of $H$. Eventually, a local asymptotic self-similarity property is proved. The limit process can be another process than fBm.

*AMS classification*: 62 G 05, 60 G 15, 60 G 17, 60 G 18.
*Keywords*: fractional Brownian motion, Gaussian processes, Hölder regularity, local asymptotic self-similarity, multi-parameter processes.

## 1 Introduction

In many applications, fractional Brownian motion (fBm) seems to fit very well to random phenomena. Recall that it can be defined by one of the four following properties. Let $H \in (0,1)$ ($H$ is sometimes called the Hurst parameter).

- $B^H$ is a centered Gaussian process such that

$$\forall s,t \in \mathbf{R}_+;\ E\left[B^H_s B^H_t\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right]$$

- the process $B^H$ such that

$$\forall t \in \mathbf{R}_+;\ B_t^H = \int_{-\infty}^{0} \left[(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right].\mathbb{W}(du) + \int_0^t (t-u)^{H-\frac{1}{2}}.\mathbb{W}(du)$$

is a fBm,

- the process $B^H$ such that

$$\forall t \in \mathbf{R}_+;\ B_t^H = \int_{\mathbf{R}} \frac{e^{it\xi}-1}{|\xi|^{H+\frac{1}{2}}}.\hat{\mathbb{W}}(d\xi)$$

is a fBm,

- $B^H$ is the unique self-similar Gaussian process with stationary increments.

Its efficiency has already been shown in simulation of traffic on Internet or in finance. This induced some recent progress such as stochastic integration against fBm.
However, the main limitation of fBm is that the Hölder regularity is constant along the paths.
Multifractional Brownian motion (mBm) has been independently introduced in [4] and [13]. This process is a generalization of fractional Brownian motion where the Hurst parameter $H$ is substituted by a function $t \mapsto H(t)$. As a consequence the Hölder exponent is allowed to vary along trajectories.
The different definitions by the two groups of authors provided two different representations of mBm.

Peltier and Levy-Vehel ([13]) defined the mBm from the moving average definition of the fractional Brownian motion

$$X_t = \int_{-\infty}^{0} \left[(t-u)^{H(t)-\frac{1}{2}} - (-u)^{H(t)-\frac{1}{2}}\right].\mathbb{W}(du) + \int_0^t (t-u)^{H(t)-\frac{1}{2}}.\mathbb{W}(du)$$

where $t \mapsto H(t)$ is a Hölder function.

Benassi, Jaffard and Roux ([4]) defined the mBm from the harmonizable representation of the fBm

$$X_t = \int_{\mathbf{R}} \frac{e^{it\xi}-1}{|\xi|^{H(t)+\frac{1}{2}}}.\hat{\mathbb{W}}(d\xi)$$

These two definitions were proved to be equivalent up to a multiplicative deterministic function ([6]).

Moreover, in [3] the covariance function of this Gaussian process has been proved to be

$$E\left[X_s X_t\right] = D\left(H(s), H(t)\right)\left[|s|^{H(s)+H(t)} + |t|^{H(s)+H(t)} - |t-s|^{H(s)+H(t)}\right]$$

where $D$ is a known deterministic function.
The goal of this paper is to study some multi-parameter extension of the multifractional Brownian motion, ie a stochastic process indexed by $\mathbf{R}_+^N$, which is an mBm when $N=1$. One extension has already been considered in [4].
2D extension of fractional Brownian motion has been already used in various applications such as underwater terrain modeling ([14]). It may be more realistic to allow local regularity to vary at each point : our extension of mBm in $\mathbf{R}^2$ may be used for this kind of application.

# 2 Multi-parameter extension of the fractional Brownian motion

Since multifractional Brownian motion is an extension of fractional Brownian motion, we start with a review of the existing extensions of fBm. Most of the results in this section are well-known, but we give new proofs based only on the covariance functions.

In the same way as Brownian motion has two main multi-parameter extensions: Levy Brownian motion and Brownian sheet, two different multi-parameter extensions of fractional Brownian motion have been defined.

## 2.1 Levy fractional Brownian motion

This process can be seen as an isotropic extension of the fractional Brownian motion. Indeed, for the fBm, we have for all $s, t \in \mathbf{R}_+$

$$E\left[X_t - X_s\right]^2 = |t-s|^{2H}$$

A natural idea to extend this process for a set of index $\mathcal{T} \subset \mathbf{R}_+^N$ is to substitute the absolute value by a norm. We get the Levy fractional Brownian motion, which is defined to be a centered Gaussian process of covariance function

$$E\left[X_s X_t\right] = \frac{1}{2}\left[\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right] \tag{1}$$

There are several definitions of this process by its trajectories. Among these, it can be defined as integral against white noise. Lindstrom stated the following (see [9]).

**Proposition 1** *The process defined by*

$$X_t = \int_{\mathbf{R}^N} \left[\|t-u\|^{H-\frac{N}{2}} - \|u\|^{H-\frac{N}{2}}\right] \mathbb{W}(du) \tag{2}$$

*is a Levy fractional Brownian motion up to a multiplicative constant.*

**Proof** This process is obviously Gaussian and centered. Thus we only have to show that the covariance function is of the form (1). We have

$$\begin{aligned} E\left[X_s - X_t\right]^2 &= \int_{\mathbf{R}^N} \left[\|t-u\|^{H-\frac{N}{2}} - \|s-u\|^{H-\frac{N}{2}}\right]^2 .du \\ &= \int_{\mathbf{R}^N} \left[\|t-s-u\|^{H-\frac{N}{2}} - \|u\|^{H-\frac{N}{2}}\right]^2 .du \end{aligned}$$

We consider the change of variables from $\mathbf{R}^N$ into itself, $v = \phi(u)$, where $\phi$ is the linear application which maps the canonic basis of $\mathbf{R}^N$ to the orthonormal basis $\left(e_1 = \frac{t-s}{\|t-s\|}, e_2, \ldots, e_N\right)$. The differential of $\phi$ in any $u \in \mathbf{R}^N$ is itself and the Jacobian

$$J\phi^{-1}_{(v_1,\ldots,v_N)} = \left|\det\left(\phi^{-1}\right)\right| = 1$$

because the matrix of $\phi$ is orthogonal.
We have

$$\begin{aligned} \|t-s-u\|^2 &= (t-s-u)^2 = \|t-s\|^2 - 2 < t-s, u > + \|u\|^2 \\ &= \|t-s\|^2 - 2\|t-s\|.v_1 + \|v\|^2 \\ &= (\|t-s\|.\epsilon_1 - v)^2 \end{aligned}$$

We obtain

$$\begin{aligned} E\left[X_s - X_t\right]^2 &= \int_{\mathbf{R}^N} \left[\| \, \|t-s\|.\epsilon_1 - v\|^{H-\frac{N}{2}} - \|v\|^{H-\frac{N}{2}}\right]^2 .dv \\ &= \|t-s\|^{2H-N} \int_{\mathbf{R}^N} \left[\|\epsilon_1 - \frac{v}{\|t-s\|}\|^{H-\frac{N}{2}} - \|\frac{v}{\|t-s\|}\|^{H-\frac{N}{2}}\right]^2 .dv \end{aligned}$$

and after the second change of variables,

$$v = \|t-s\|.w = \|t-s\| Id.w$$

we get

$$E\left[X_s - X_t\right]^2 = \|t-s\|^{2H} \underbrace{\int_{\mathbf{R}^N} \left[\|\epsilon_1 - w\|^{H-\frac{N}{2}} - \|w\|^{H-\frac{N}{2}}\right]^2 .dw}_{K_{N,H}}$$

therefore

$$E\left[X_s X_t\right] = K_{N,H}\left[\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right]$$

□

The harmonizable representation of fractional Brownian motion can also be generalized. Before that, let's recall briefly definitions of white noise and its Fourier transform.
In the following, we will denote $L^2_{\mathbf{C}}(\mathbf{R}^N)$ the set of functions $f : \mathbf{R}^N \to \mathbf{C}$ such that $\int_{\mathbf{R}^N} |f(u)|^2 du < \infty$.

**Definition 1** *The complex isonormal process is defined to be a centered Gaussian process* $W = \left\{W(f); f \in L^2_{\mathbf{C}}(\mathbf{R}^N)\right\}$ *such that*

$$\forall f, g \in L^2_{\mathbf{C}}(\mathbf{R}^N);\ E\left[W(f)\overline{W(g)}\right] = \int_{\mathbf{R}^N} f(u)\overline{g(u)}.du$$

*Then, white noise* $\mathbb{W}$ *can be defined by*

$$\mathbb{W}(E) = W\left(\mathbb{1}_E\right)$$

**Definition 2** *A Gaussian process* $\left\{\hat{W}(f); f \in L^2_{\mathbf{C}}(\mathbf{R}^N)\right\}$ *is said to be the Fourier transform of a complex isonormal process* $\left\{W(f); f \in L^2_{\mathbf{C}}(\mathbf{R}^N)\right\}$ *if for all* $f \in L^2_{\mathbf{C}}(\mathbf{R}^N)$

$$\hat{W}(f) = W(\hat{f})$$

*where* $\hat{f}$ *if the Fourier transform of the function* $f$.
*The Fourier transform of white noise is defined in the same way.*

This complex measure is usually used to define the harmonizable representation of fractional Brownian motion

$$B_t^H = \int_{\mathbf{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+\frac{1}{2}}}.\hat{\mathbb{W}}(d\xi)$$

that can be generalized in the following.

**Proposition 2** *The process defined by*

$$X_t = \int_{\boldsymbol{R}^N} \frac{e^{i\langle t,\xi\rangle} - 1}{\|\xi\|^{H+\frac{N}{2}}}.\hat{\mathbb{W}}(d\xi) \tag{3}$$

*where* $\hat{\mathbb{W}}$ *is the Fourier transform of white noise in* $\boldsymbol{R}^N$*,*
*is a Levy fractional Brownian motion up to a multiplicative constant.*

**Proof** As will be done for multifractional Brownian field, the Fourier transform of the kernel of representation (2) could be directly computed. But as this representation defines a real centered Gaussian process, it is enough to show that the covariance function has the form (1).
For all $t \in \mathbf{R}^N$, let's denote by $f_t$ the function $\xi \mapsto \frac{e^{i<t,\xi>}-1}{\|\xi\|^{H+\frac{N}{2}}}$ and consider the centered Gaussian process $X = \left\{X_t = \hat{W}(f_t); t \in \mathbf{R}_+^N\right\}$.
First of all, let's show that, almost surely, $\hat{W}(f_t) \in \mathbf{R}$.
In fact, using $\hat{W}(f_t) = W(\hat{f}_t)$, showing that $\hat{f}_t \in \mathbf{R}$ is sufficient.
Indeed, by $E\left[Im\left(W(f)\right)\right]^2 = \int_{\mathbf{R}^N} \left(Im\left(f\right)\right)^2 = 0$, $f \in \mathbf{R}$ imply $W(f) \in \mathbf{R}$ almost surely, and

$$\begin{aligned} Im(\hat{f}_t(x)) &= \int_{\mathbf{R}^N} Im\left(e^{-i<x,u>} f_t(u)\right).du \\ &= \int_{\mathbf{R}^N} \frac{\sin < t-x, u >}{\|u\|^{H+\frac{N}{2}}}.du \\ &= 0 \end{aligned}$$

by parity.
The process $X$ is therefore real and its covariance function is

$$\begin{aligned} E\left[X_s X_t\right] &= E\left[\hat{W}(f_s)\overline{\hat{W}(f_t)}\right] \\ &= \int_{\mathbf{R}^N} \frac{\left(e^{i<s,\xi>} - 1\right)\left(e^{-i<t,\xi>} - 1\right)}{\|\xi\|^{2H+N}}.d\xi \\ &= \int_{\mathbf{R}^N} \frac{e^{i<s-t,\xi>} - e^{i<s,\xi>} - e^{-i<t,\xi>} + 1}{\|\xi\|^{2H+N}}.d\xi \end{aligned}$$

Then we have to consider 3 integrals of the form $\int_{\mathbf{R}^N} \frac{1-e^{i<t,\xi>}}{\|\xi\|^{2H+N}}.d\xi$.
As in proposition 1, for $t \in \mathbf{R}^N$ fixed, consider the change of variables from $\mathbf{R}^N$ into itself, $u = \phi(\xi)$ where $\phi$ is the linear application which maps the canonic

basis of $\mathbf{R}^N$ to the orthonormal basis $\left(e_1 = \frac{t}{||t||}, e_2, \ldots, e_N\right)$.
Then, we get

$$\int_{\mathbf{R}^N} \frac{1 - e^{i<t,\xi>}}{||\xi||^{2H+N}}.d\xi = \int_{\mathbf{R}^N} \frac{1 - e^{i||t||.u_1}}{||u||^{2H+N}}.du$$

After the second change of variables

$$v = ||t||.u = ||t||Id.u$$
$$dv = ||t||^N.du$$

we get

$$\int_{\mathbf{R}^N} \frac{1 - e^{i<t,\xi>}}{||\xi||^{2H+N}}.d\xi = \frac{||t||^{2H+N}}{||t||^N} \underbrace{\int_{\mathbf{R}^N} \frac{1 - e^{iv_1}}{||v||^{2H+N}}.dv}_{C_{N,H}>0}$$

Proceeding the same way for the 2 other integrals, we can conclude

$$E\left[X_s X_t\right] = C_{N,H} \left[||s||^{2H} + ||t||^{2H} - ||t-s||^{2H}\right]$$

which shows that the process $\left\{\frac{1}{\sqrt{C_{N,H}}}\hat{W}(f_t), t \in \mathbf{R}_+^N\right\}$ is a Levy fractional Brownian motion. □

## 2.2 Fractional Brownian sheet

On the contrary to the Levy fractional Brownian motion, this process is not isotropic. In particular, we can have different Hurst parameters in each of the $N$ directions.
For the fBm, we have for all $s, t \in \mathbf{R}_+$

$$E\left[X_s X_t\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right]$$

As in the definition of Brownian sheet, another way to generalize fBm is to set the covariance equal to the tensor product of one dimensional covariances. Then, fractional Brownian sheet (fBs) is defined to be a centered Gaussian process of covariance function

$$E\left[X_s X_t\right] = \prod_{i=1}^{N} \frac{1}{2}\left(s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}\right) \tag{4}$$

As in the isotropic case, this process has two different representations by its trajectories.

**Proposition 3** *The process defined by*

$$X_t = \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left[|t_i - u_i|^{H_i - \frac{1}{2}} - |u_i|^{H_i - \frac{1}{2}}\right] \mathbb{W}(du)$$

*is a fractional Brownian sheet, up to a multiplicative constant.*

**Remark 1** *In [8], Pontier/Leger introduced another moving average representation of fractional Brownian sheet.*

$$X_t = \int_{\boldsymbol{R}^N} \prod_{i=1}^{N} \left[ (t_i - u_i)_+^{H_i - \frac{1}{2}} - (-u_i)_+^{H_i - \frac{1}{2}} \right] \mathbb{W}(du)$$

**Proof** This process is obviously Gaussian and centered. Thus, we only need to show that its covariance function has the expected form. We compute

$$\begin{aligned} E\left[X_s X_t\right] &= \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left[ |s_i - u_i|^{H_i - \frac{1}{2}} - |u_i|^{H_i - \frac{1}{2}} \right] \left[ |t_i - u_i|^{H_i - \frac{1}{2}} - |u_i|^{H_i - \frac{1}{2}} \right] .du \\ &= \prod_{i=1}^{N} \int_{\mathbf{R}} \left[ |s_i - u_i|^{H_i - \frac{1}{2}} - |u_i|^{H_i - \frac{1}{2}} \right] \left[ |t_i - u_i|^{H_i - \frac{1}{2}} - |u_i|^{H_i - \frac{1}{2}} \right] .du_i \end{aligned}$$

We can see that the factor corresponding to each $i$, is the covariance of a fBm with Hurst parameter $H_i$ (or a Levy fractional Brownian motion with $N = 1$). Then we have

$$E\left[X_s X_t\right] = \prod_{i=1}^{N} K_{1,H_i} \left[ |s_i|^{2H_i} + |t_i|^{2H_i} - |t_i - s_i|^{2H_i} \right]$$

□

This process also has an harmonizable representation, using the Fourier transform of the white noise in $\mathbf{R}^N$ as in the previous paragraph.

**Proposition 4** *For all $t = (t_i)$, consider the function $\phi_t$ such that for all $\xi = (\xi_i)$,*

$$\phi_t(u) = \prod_{m=1}^{N} \frac{e^{it_m \xi_m} - 1}{|\xi_m|^{H_m + \frac{1}{2}}}$$

*The process defined by*

$$X_t = \hat{W}\left(\phi_t\right) = \int_{\boldsymbol{R}^N} \prod_{m=1}^{N} \frac{e^{it_m \xi_m} - 1}{|\xi_m|^{H_m + \frac{1}{2}}} \hat{\mathbb{W}}(d\xi)$$

*is a fractional Brownian sheet, up to a multiplicative constant.*

**Proof** As in the previous proposition, let's compute the covariance function of this process.

$$\begin{aligned} E\left[X_s X_t\right] &= \int_{\mathbf{R}^N} \prod_{m=1}^{N} \frac{\left(e^{is_m \xi_m} - 1\right)\left(e^{-it_m \xi_m} - 1\right)}{|\xi_m|^{2H_m + 1}} .d\xi \\ &= \prod_{m=1}^{N} \int_{\mathbf{R}} \frac{\left(e^{is_m \xi_m} - 1\right)\left(e^{-it_m \xi_m} - 1\right)}{|\xi_m|^{2H_m + 1}} .d\xi_m \\ &= \prod_{m=1}^{N} C_{1,H_m} \left[ |s_m|^{2H_m} + |t_m|^{2H_m} - |t_m - s_m|^{2H_m} \right] \end{aligned}$$

using the same argument of the previous proposition. □

**Remark 2** *The processes defined in propositions 3 and 4 are proved to have the same law. In fact, as a particular case of proposition 10, they are indistinguishable.*

## 2.3 Stationarity of increments and self similarity

Let us start by recalling the notion of increments in $\mathbf{R}_+^N$.
For a function $f : [0,1]^N \to \mathbf{R}$ and $h \in \mathbf{R}$, one usually define the progressive difference in direction $\epsilon_i$ by

$$\Delta_{h,i} f(x) = \begin{cases} f(x + h\epsilon_i) - f(x) & \text{if } x, x + h\epsilon_i \in [0,1]^N \\ 0 & \text{either} \end{cases}$$

and for $h \in \mathbf{R}^N$ and $A = (i_1, \ldots, i_k)$,

$$\Delta_{h,A} f = \Delta_{h_{i_1}, i_1} f \circ \cdots \circ \Delta_{h_{i_k}, i_k} f$$

Despite the temptation to define the increments by $X_t - X_s$ as in one dimension, it is better to set

$$\begin{aligned} \Delta X_{s,t} &= \Delta_{t-s,(1,\ldots,N)} X_s \\ &= \sum_{r \in \{0,1\}^N} (-1)^{N - \sum_l r_l} X_{[s_i + r_i(t_i - s_i)]_i} \end{aligned} \tag{5}$$

If there exists $i \in \{1, \ldots, N\}$ such that $s_i = t_i$, we have $\Delta X_{s,t} = 0$. Then, we consider

$$I = \{i = 1, \ldots, N;\ s_i \neq t_i\}$$

and

$$\Delta_{t-s,I} X_s = \sum_{r \in \{0,1\}^{\#I}} (-1)^{\#I - \sum_l r_l} X_{[s_i + r_i(t_i - s_i)]_{i \in I}}$$

### 2.3.1 Isotropic case

In the isotropic case, the following extension of fBm's properties are well known (see [9]).

**Proposition 5** *Let $X = \left\{X_t; t \in \mathbf{R}_+^N\right\}$ be a Levy fractional Brownian motion. We have the two following properties for all $h \in \mathbf{R}_+^N$ and $a > 0$*

$$\begin{aligned} X_{t+h} - X_h &\overset{(d)}{=} X_t - X_0 \\ X_{at} &\overset{(d)}{=} a^H X_t \end{aligned}$$

*where $\overset{(d)}{=}$ means equality of finite dimensional distributions.*

**Proof** For all $s$ and $t$ in $\mathbf{R}_+^N$, we have

$$\begin{aligned} E\left[(X_{s+h} - X_h)(X_{t+h} - X_h)\right] &= \frac{1}{2}\left(E\left[X_{s+h} - X_h\right]^2 + E\left[X_{t+h} - X_h\right]^2 - E\left[X_{t+h} - X_{s+h}\right]^2\right) \\ &= \frac{1}{2}\left(\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right) \\ &= E\left[X_s X_t\right] \end{aligned}$$

For self-similarity, we compute

$$\begin{aligned} E\left[X_{as}X_{at}\right] &= \frac{1}{2}\left(E\left[X_{as}\right]^2 + E\left[X_{at}\right]^2 - E\left[X_{at}-X_{as}\right]^2\right) \\ &= \frac{1}{2}\left(\|as\|^{2H} + \|at\|^{2H} - \|at-as\|^{2H}\right) \\ &= a^{2H}E\left[X_sX_t\right] \\ &= E\left[a^HX_s\, a^HX_t\right] \end{aligned}$$

□

Proposition 5 implies the stationarity of increments (5).

**Proposition 6** *The increments of Levy fractional Brownian are stationary, ie for all* $h \in \boldsymbol{R}_+^N$

$$\Delta X_{h,t+h} \overset{(d)}{=} \Delta X_{0,t}$$

**Proof** We fix $h \in \mathbf{R}_+^N$ and write

$$\Delta X_{h,t+h} = \sum_{r\in\{0,1\}^N-\{0\}} (-1)^{N-\sum_l r_l}\left(X_{[h_i+r_it_i]_i} - X_h\right)$$

then in the development of $E\left[\Delta X_{h,s+h}\Delta X_{h,t+h}\right]$, we only have terms of the form

$$E\left[\left(X_{[h_i+r_is_i]_i} - X_h\right)\left(X_{[h_i+\rho_it_i]_i} - X_h\right)\right] = E\left[X_{[r_is_i]_i}X_{[\rho_it_i]_i}\right]$$

using the previous proposition. Therefore we have

$$E\left[\Delta X_{h,s+h}\Delta X_{h,t+h}\right] = E\left[\Delta X_{0,s}\Delta X_{0,t}\right]$$

□

### 2.3.2 Non-isotropic case

In the non-isotropic case, the properties of self-similarity and stationarity of increments have been stated by Léger/Pontier (cf [8]). Here, we give another proof based on the covariance function rather than the moving average representation.

**Proposition 7** *Let* $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ *be a fractional Brownian sheet. We have the two following properties for all* $h \in \boldsymbol{R}_+^N$ *and* $a > 0$

$$\Delta X_{h,t+h} \overset{(d)}{=} \Delta X_{0,t}$$

$$X_{at} \overset{(d)}{=} a^{\sum_i H_i}\, X_t$$

**Proof** We consider $N$ independent fBm $X^{(1)}, \ldots X^{(N)}$ of Hurst parameter $H_i$, and the process $Y = \left\{Y_t; t \in \mathbf{R}_+^N\right\}$ such that $Y_t = \prod_{i=1}^N X_{t_i}^{(i)}$. We can see easily that $X$ and $Y$ have the same covariance function. The same result follows for the increments $\left\{\Delta X_{h,t+h}; t \in \mathbf{R}_+^N\right\}$ and $\left\{\Delta Y_{h,t+h}; t \in \mathbf{R}_+^N\right\}$. As a consequence, from

$$\begin{aligned}\Delta Y_{h,t+h} &= \sum_{r \in \{0,1\}^N} (-1)^{N - \sum_l r_l} \prod_{i=1}^N X_{h_i + r_i t_i}^{(i)} \\ &= \prod_{i=1}^N \left[X_{t_i+h_i}^{(i)} - X_{h_i}^{(i)}\right]\end{aligned}$$

then we have

$$\begin{aligned}E\left[\Delta X_{h,s+h} \Delta X_{h,t+h}\right] &= E\left[\Delta Y_{h,s+h} \Delta Y_{h,t+h}\right] \\ &= \prod_{i=1}^N \underbrace{E\left[\left(X_{s_i+h_i}^{(i)} - X_{h_i}^{(i)}\right)\left(X_{t_i+h_i}^{(i)} - X_{h_i}^{(i)}\right)\right]}_{E\left[X_{s_i}^{(i)} X_{t_i}^{(i)}\right]} \\ &= E\left[\Delta X_{0,s} \Delta X_{0,t}\right]\end{aligned}$$

For self-similarity, we compute for all $a > 0$

$$\begin{aligned}E\left[X_{as} X_{at}\right] &= \prod_{i=1}^N \frac{1}{2}\left((as_i)^{2H_i} + (at_i)^{2H_i} - |at_i - as_i|^{2H_i}\right) \\ &= a^{2\sum_i H_i} \prod_{i=1}^N \frac{1}{2}\left(s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}\right) \\ &= a^{2\sum_i H_i} E\left[X_s X_t\right] \\ &= E\left[a^{\sum_i H_i} X_s \, a^{\sum_i H_i} X_t\right]\end{aligned}$$

□

Therefore, we can conclude that both extensions of fBm satisfy the properties of self-similarity and stationarity of increments.

# 3 The multifractional Brownian motion's case

Once again, we can consider two different kinds of multi-parameter extension of mBm : isotropic and anisotropic extension. Note, first of all, that mBm already has a multi-parameter extension. Indeed, the formulation of Benassi/Jaffard/Roux in [4] was done for $t \in \mathbf{R}^N$. We will see that it can be considered as an isotropic extension.

## 3.1 Isotropic extension

To define an isotropic extension of the mBm, the natural way is to substitute the constant $H$ of the moving average representation of the Levy fractional Brownian motion, with a function.

**Definition 3** *Let $H : R^N \to (0,1)$ be a measurable function. The process $\left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ such that*

$$X_t = \int_{\boldsymbol{R}^N} \left[ ||t-u||^{H(t)-\frac{N}{2}} - ||u||^{H(t)-\frac{N}{2}} \right] \mathbb{W}(du) \tag{6}$$

*is called multifractional Brownian field.*

We will show that this process *is the same* as the process defined by Benassi/Jaffard/Roux. This result generalizes on the equivalence stated in the case $N = 1$ in [6].

**Proposition 8** *Let $H : R^N \to (0,1)$ be a measurable function. The process defined by*

$$X_t = \int_{R^N} \frac{e^{i\langle t,\xi\rangle} - 1}{||\xi||^{H(t)+\frac{N}{2}}} . \hat{\mathbb{W}}(d\xi) \tag{7}$$

*is indistinguishable, up to a multiplicative deterministic function, from the process defined by (6). This formulation is the harmonizable representation of the multifractional Brownian field.*

**Proof** First of all, let us compute the Fourier transform of the function $||.||^\alpha$.

$$\begin{aligned} \langle \mathcal{T}||.||^\alpha, \varphi \rangle &= \langle ||.||^\alpha, \hat{\varphi} \rangle \\ &= \int_{\mathbf{R}^N} ||t||^\alpha \left( \int_{\mathbf{R}^N} e^{-i<w,t>} \varphi(w).dw \right) .dt \end{aligned}$$

we consider the change of variables

$$\begin{aligned} \mathbf{R}^N \times \mathbf{R}^N &\to \mathbf{R}^N \times \mathbf{R}^N \\ (w,t) &\mapsto (w, \lambda = \phi(t)) \end{aligned}$$

where $\phi$ is the linear application which maps the canonic basis of $\mathbf{R}^N$ to the orthonormal basis $\left(e_1 = \frac{w}{||w||}, e_2, \ldots, e_N\right)$. We get

$$\begin{aligned} \langle \mathcal{T}||.||^\alpha, \varphi \rangle &= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} ||\lambda||^\alpha e^{i\lambda_1 ||w||} \varphi(w).dw.d\lambda \\ &= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \frac{||u||^\alpha}{||w||^\alpha} e^{-iu_1} \varphi(w) \frac{dw.du}{||w||^N} \end{aligned}$$

using the change of variables $(w,\lambda) \mapsto (w, u = ||w||\lambda)$. Then we have

$$\langle \mathcal{T}||.||^\alpha, \varphi \rangle = \underbrace{\left( \int_{\mathbf{R}^N} ||u||^\alpha e^{-iu_1}.du \right)}_{\lambda_\alpha} \int_{\mathbf{R}^N} \frac{1}{||w||^{\alpha+N}} \varphi(w).dw$$

Thus,

$$\mathcal{T}||.||^\alpha(w) = \frac{\lambda_\alpha}{||w||^{\alpha+N}}$$

We use this result to calculate the Fourier transform of $||t-.||^\alpha - ||.||^\alpha$. We will use the following property : if $g(u)=f(u-\alpha)$ then $\hat{g}=e^{-i<\alpha,v>}\hat{f}(v)$.

$$\mathcal{T}\left[||t-.||^\alpha - ||.||^\alpha\right](v) = \left[e^{-i<t,v>}-1\right]\frac{\lambda_\alpha}{||v||^{\alpha+N}}$$

We deduce from this

$$\mathcal{T}\left[||t-.||^{H(t)-\frac{N}{2}} - ||.||^{H(t)-\frac{N}{2}}\right](v) = \lambda_{H(t)}\overline{\left(\frac{e^{i<t,v>}-1}{||v||^{H(t)+\frac{N}{2}}}\right)}$$

and $\forall t \in \mathbf{R}^N$, we have almost surely

$$\int_{\mathbf{R}^N}\left[||t-u||^{H(t)-\frac{N}{2}} - ||u||^{H(t)-\frac{N}{2}}\right]\mathbb{W}(du) = \lambda_{H(t)}\int_{\mathbf{R}^N}\frac{e^{i\langle t,\xi\rangle}-1}{||\xi||^{H(t)+\frac{N}{2}}}.\hat{\mathbb{W}}(d\xi)$$

using the fact we saw previously that the second integral is almost surely real. Therefore, by an argument of continuity, the result follows. □

This process is obviously a centered Gaussian process. It is thus of interest to study its covariance function. The following proposition is an extension of the case $N=1$ stated in [3].

**Proposition 9** *Let* $\left\{X_t; t\in \boldsymbol{R}_+^N\right\}$ *be a multifractional Brownian field. There exists a deterministic function* $D_N^f : \boldsymbol{R}\to\boldsymbol{R}$ *such that the covariance function of* $X$ *can be written*

$$E\left[X_sX_t\right] = D_N^f(H(s)+H(t))\left[||s||^{H(s)+H(t)} + ||t||^{H(s)+H(t)} - ||t-s||^{H(s)+H(t)}\right] \tag{8}$$

**Proof** The easiest way to show this result is to use the harmonizable representation. By definition of $\hat{\mathbb{W}}$, we have

$$E\left[X_sX_t\right] = \int_{\mathbf{R}^N}\frac{\left(e^{i<s,\xi>}-1\right)\left(e^{-i<t,\xi>}-1\right)}{||\xi||^{H(s)+H(t)+N}}.d\xi$$

This integral has already been calculated for a Levy fractional Brownian motion with a parameter $H=\frac{H(s)+H(t)}{2}$. Then we have

$$E\left[X_sX_t\right] = \underbrace{\left(\int_{\mathbf{R}^N}\frac{1-e^{iu_1}}{||u||^{H(s)+H(t)+N}}.du\right)}_{D_N^f(H(s)+H(t))}\left[||s||^{H(s)+H(t)} + ||t||^{H(s)+H(t)} - ||t-s||^{H(s)+H(t)}\right]$$

with $D_N^f(x) = \int_{\mathbf{R}^N}\frac{1-e^{iu_1}}{||u||^{x+N}}.du$ □

## 3.2 Non isotropic extension

Another way to extend the multifractional Brownian motion for a set of index included in $\mathbf{R}_+^N$, is to copy the definition of the Brownian sheet.

**Definition 4** *Let $H : \boldsymbol{R}_+^N \to (0,1)^N$ be a measurable function. The process $\left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ such that*

$$X_t = \int_{\boldsymbol{R}^N} \prod_{i=1}^{N} \left[|t_i - u_i|^{H_i(t)-\frac{1}{2}} - |u_i|^{H_i(t)-\frac{1}{2}}\right] \mathbb{W}(du)$$

*where $\mathbb{W}$ is the white noise, is called multifractional Brownian sheet (mBs).*

As in the case of the isotropic extension, there also exists a harmonizable representation of the mBs.

**Proposition 10** *Let $H : \boldsymbol{R}_+^N \to (0,1)^N$ be a measurable function. For all $t = (t_i)_{i\in\{1;\ldots;N\}}$, we consider the function $\phi_t$ such that for all $\xi = (\xi_i)$,*

$$\phi_t(u) = \prod_{m=1}^{N} \frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m(t)+\frac{1}{2}}}$$

*The process defined by*

$$X_t = \hat{W}\left(\phi_t\right) = \int_{\boldsymbol{R}^N} \prod_{m=1}^{N} \frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m(t)+\frac{1}{2}}} \hat{\mathbb{W}}(d\xi)$$

*is indistinguishable, up to a multiplicative deterministic function, from the process defined previously. This formulation is the harmonizable representation of the multifractional Brownian sheet.*

**Proof** We have already seen that for each $m \in \{1,\ldots,N\}$

$$\mathcal{T}\left[|t_m - .|^{H_m(t)-\frac{1}{2}} - |.|^{H_m(t)-\frac{1}{2}}\right](\xi_m) = \lambda_{H_m(t)}\overline{\left(\frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m(t)+\frac{1}{2}}}\right)}$$

Moreover, we compute

$$\begin{aligned}
\mathcal{T}\left(\prod_{m=1}^{N}\left[|t_m - .|^{H_m(t)-\frac{1}{2}} - |.|^{H_m(t)-\frac{1}{2}}\right]\right) & (\xi) \\
&= \int_{\mathbf{R}^N} e^{-i<\xi,x>} \prod_{m=1}^{N}\left[|t_m - x_m|^{H_m(t)-\frac{1}{2}} - |x_m|^{H_m(t)-\frac{1}{2}}\right].dx \\
&= \int_{\mathbf{R}^N} \prod_{m=1}^{N} e^{-i\xi_m x_m}\left[|t_m - x_m|^{H_m(t)-\frac{1}{2}} - |x_m|^{H_m(t)-\frac{1}{2}}\right].dx \\
&= \prod_{m=1}^{N} \mathcal{T}\left[|t_m - .|^{H_m(t)-\frac{1}{2}} - |.|^{H_m(t)-\frac{1}{2}}\right](\xi_m)
\end{aligned}$$

Therefore

$$\underbrace{\left(\prod_{i=1}^{N}\lambda_m(t)\right)}_{\lambda(t)} \hat{W}\overline{\left(\prod_{m=1}^{N}\frac{e^{it_m\cdot} - 1}{|.|^{H_m(t)+\frac{1}{2}}}\right)} = W\left(\prod_{m=1}^{N}\left[|t_m - .|^{H_m(t)-\frac{1}{2}} - |.|^{H_m(t)-\frac{1}{2}}\right]\right)$$

We use the same arguments as in proposition 8 to conclude. □

The following proposition shows that the covariance structure of multifractional Brownian sheet, is a generalization of the fBs's one.

**Proposition 11** *Let* $\left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ *be a multifractional Brownian sheet. There exists a deterministic function* $D^s : \boldsymbol{R}^N \to \boldsymbol{R}$ *such that*

$$E\left[X_s X_t\right] = D^s\left(H(s) + H(t)\right) \prod_{m=1}^{N} \left[|s_m|^{H_m(s)+H_m(t)} + |t_m|^{H_m(s)+H_m(t)} - |t_m - s_m|^{H_m(s)+H_m(t)}\right] \tag{9}$$

**Proof** As usually, we use the harmonizable representation of the process

$$\begin{aligned} E\left[X_s X_t\right] &= \int_{\mathbf{R}^N} \prod_{m=1}^{N} \frac{\left(e^{i s_m \xi_m} - 1\right)\left(e^{-i t_m \xi_m} - 1\right)}{|\xi_m|^{H_m(s)+H_m(t)+1}} . d\xi \\ &= \prod_{m=1}^{N} \int_{\mathbf{R}} \frac{\left(e^{i s_m \xi_m} - 1\right)\left(e^{-i t_m \xi_m} - 1\right)}{|\xi_m|^{H_m(s)+H_m(t)+1}} . d\xi_m \end{aligned}$$

We remark that the factor corresponding to each $m$, is the covariance of a multifractional Brownian motion, with has already been calculated. Therefore we have

$$E\left[X_s X_t\right] = \prod_{m=1}^{N} D_1^f\left(H_m(s) + H_m(t)\right)\left[|s_m|^{H_m(s)+H_m(t)} + |t_m|^{H_m(s)+H_m(t)} - |t_m - s_m|^{H_m(s)+H_m(t)}\right]$$

□

**Remark 3** *The form of the previous covariance function gives the idea to consider the process* $Y = \left\{Y_t; t \in \boldsymbol{R}_+^N\right\}$ *defined from* $N$ *independent multifractional Brownian motions* $X^{(i)}$ *with parameter* $H_i$ *by*

$$Y_t = X^{(1)}_{t^{(1)}} \ldots X^{(N)}_{t^{(N)}}$$

*Although* $Y$ *is not a Gaussian process, it is easily seen that it has the same covariance function as a multifractional Brownian sheet. This remark will be often used in the following.*

# 4 Regularity

A lot of properties are known about the regularity of the trajectories of Brownian motion and fractional Brownian motion. As we will see, in the case of the multiparameter extension of the mBm, we have to make some assumptions about the regularity of $H$ before studying the continuity of trajectories. In the definitions of mBm (cf [1] and [4]), the function $H$ is supposed to be Hölder continuous.

## 4.1 Continuity of the two extensions

We first recall the Kolmogorov's criterion.

**Theorem 1 (Kolmogorov)** *Consider a process* $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ *such that there exists* $C > 0$*,* $p > 0$ *and* $\gamma > N$

$$\forall s, t \in \boldsymbol{R}_+^N,\ E\left[|X_t - X_s|^p\right] \leq C\|t-s\|^\gamma \tag{10}$$

*Then, there exists a modification* $Y = \left\{Y_t; t \in \boldsymbol{R}_+^N\right\}$ *of* $X$ *that is Hölder continuous of any order* $q \in (0, \frac{\gamma-N}{p})$*.*

As usually, the quantity $E\left[|X_t - X_s|^2\right]$ is studied for $s, t \in [a, b]$ where $a \preceq b$ and then, a patching argument is used to extend to $s, t \in \mathbf{R}_+^N$.

### 4.1.1 Isotropic case

**Lemma 1** *For all* $\eta$ *and* $\mu$ *such that* $0 < \eta < \mu < 1$*, the multiplicative factor* $D_N^f$ *of covariance function in (9), is positive and belongs to* $C^\infty([\eta, \mu])$*. Moreover, its order* $n$ *derivative is given by*

$$D_N^{f\,(n)}(x) = \int_{\boldsymbol{R}^N} \frac{1 - e^{iu_1}}{\|u\|^{x+N}} \ln^n \frac{1}{\|u\|}.du \tag{11}$$

**Proof** As the integral of a positive function, $D_N^f$ is positive. By an argument of uniform convergence of integrals (11) on $[\eta, \mu]$, $D_N^f$ is $C^\infty([\eta, \mu])$ and the derivatives are obtained by derivations of the integrand. □

**Proposition 12** *For all* $s, t \in [a, b]$*, we have*

$$\begin{aligned}
\frac{1}{2}E\left[X_t - X_s\right]^2 &= D\left[H(s)+H(t)\right] \times \|t-s\|^{H(s)+H(t)} \\
&+\frac{1}{2}\left[\frac{\partial^2\varphi}{\partial x^2}\left(H(s)+H(t);\|s\|\right) + \frac{\partial^2\varphi}{\partial x^2}\left(H(s)+H(t);\|t\|\right)\right] \times \left(H(t)-H(s)\right)^2 \\
&+O_{a,b}\left[\left(H(t)-H(s)\right)\left(\|t\|-\|s\|\right)\right] + o_{a,b}\left(H(t)-H(s)\right)^2
\end{aligned} \tag{12}$$

*where* $\varphi(x, y) = D(x)y^x$*.*

**Proof** Using the covariance function of the multifractional Brownian field, we have

$$\begin{aligned}
\frac{1}{2}E\left[|X_s - X_t|^2\right] &= D\left[2H(s)\right]\ \|s\|^{2H(s)} - D\left[H(s)+H(t)\right]\ \|s\|^{H(s)+H(t)} \\
&+D\left[2H(t)\right]\ \|t\|^{2H(t)} - D\left[H(s)+H(t)\right]\ \|t\|^{H(s)+H(t)} \\
&+D\left[H(s)+H(t)\right]\ \|t-s\|^{H(s)+H(t)}
\end{aligned} \tag{13}$$

We have to get a second order expansion of this expression.
We introduce the function $\varphi$ defined by

$$\varphi(x, y) = D(x)y^x$$

We can write

$$\frac{1}{2}E\left[|X_s - X_t|^2\right] = \varphi(2H(s), \|s\|) - \varphi(H(s)+H(t), \|s\|) + \varphi(2H(t), \|t\|) - \varphi(H(s)+H(t), \|t\|) + D\left[H(s)+H(t)\right] \|t-s\|^{H(s)+H(t)} \tag{14}$$

We use the second order expansion

$$\begin{aligned}\varphi(2H(s), \|s\|) - \varphi(H(s)+H(t), \|s\|) &= (H(s)-H(t)) \times \frac{\partial\varphi}{\partial x}(H(s)+H(t), \|s\|) \\ &+ \frac{(H(s)-H(t))^2}{2} \times \frac{\partial^2\varphi}{\partial x^2}(H(s)+H(t), \|s\|) \\ &+ o_{a,b}(H(s)-H(t))^2\end{aligned}$$

An inversion of roles between $s$ and $t$ provides the expansion of

$$\varphi(2H(t), \|t\|) - \varphi(H(s)+H(t), \|t\|)$$

Then (14) becomes

$$\begin{aligned}\frac{1}{2}E\left[|X_s - X_t|^2\right] &= (H(t)-H(s)) \times \left[\frac{\partial\varphi}{\partial x}(H(s)+H(t), \|t\|) - \frac{\partial\varphi}{\partial x}(H(s)+H(t), \|s\|)\right] \\ &+ \frac{(H(t)-H(s))^2}{2} \times \left[\frac{\partial^2\varphi}{\partial x^2}(H(s)+H(t), \|s\|) + \frac{\partial^2\varphi}{\partial x^2}(H(s)+H(t), \|t\|)\right] \\ &+ D\left[H(s)+H(t)\right] \|t-s\|^{H(s)+H(t)} + o_{a,b}(H(t)-H(s))^2\end{aligned}$$

Since

$$(H(t)-H(s)) \times \left[\frac{\partial\varphi}{\partial x}(H(s)+H(t), \|t\|) - \frac{\partial\varphi}{\partial x}(H(s)+H(t), \|s\|)\right]$$

is $O_{a,b}\left[(H(t)-H(s))(\|t\| - \|s\|)\right]$, the result follows. □

**Corollary 1** *For all $s, t \in [a, b]$, we have*

$$\begin{aligned}\frac{1}{2}E\left[X_t - X_s\right]^2 &= D\left[2H(t)\right] \times \|t-s\|^{2H(t)} \\ &+ \frac{\partial^2\varphi}{\partial x^2}(2H(t); \|t\|) \times (H(t)-H(s))^2 \\ &+ o_{a,b}(H(t)-H(s))^2 + o_{a,b}\left(\|t-s\|^{2H(t)}\right)\end{aligned} \tag{15}$$

*where $\varphi(x, y) = D(x)y^x$.*

**Proof** Using the expansion of $D\left[H(s)+H(t)\right]$ and

$$\|t-s\|^{H(s)+H(t)} = \|t-s\|^{2H(t)} - (H(t)-H(s))\|t-s\|^{2H(t)} \ln\|t-s\| + o_{a,b}(H(t)-H(s))^2$$

we get

$$\begin{aligned}D\left[H(s)+H(t)\right] \times \|t-s\|^{H(s)+H(t)} &= D\left[2H(t)\right] \times \|t-s\|^{2H(t)} \\ &+ o_{a,b}\left(\|t-s\|^{2H(t)}\right) + o_{a,b}(H(t)-H(s))^2\end{aligned} \tag{16}$$

Moreover as $H(t) < 1$ for all $t \in [a,b]$, we have $\epsilon = 1 - H(t) > 0$ and

$$\begin{aligned} 2\left(H(t)-H(s)\right)\left(\|t\|-\|s\|\right) &= 2\left(H(t)-H(s)\right)\left(\|t\|-\|s\|\right)^{\frac{\epsilon}{2}} \times \left(\|t\|-\|s\|\right)^{1-\frac{\epsilon}{2}} \\ &\leq \left(H(t)-H(s)\right)^2\left(\|t\|-\|s\|\right)^{\epsilon} + \left(\|t\|-\|s\|\right)^{2-\epsilon} \end{aligned}$$

that implies

$$\left(H(t)-H(s)\right)\left(\|t\|-\|s\|\right) = o_{a,b}\left(H(t)-H(s)\right)^2 + o_{a,b}\left(\|t-s\|^{2H(t)}\right) \tag{17}$$

We conclude by (12), (16) and (17) using first order expansion of $\frac{\partial^2\varphi}{\partial x^2}$ in $x$ and $y$. □

Using the continuity of $D$, $D'$ and $D''$, we can state from the previous proposition

**Corollary 2** *There exist positive constants $K$ and $L$ such that*

$$\forall s,t \in [a,b];\ E\left[X_t - X_s\right]^2 \leq K\ \|t-s\|^{2H(t)} + L\ |H(t)-H(s)|^2 \tag{18}$$

**Corollary 3** *Suppose $H$ is $\beta$-Hölder continuous. There exists a constant $M$ such that*

$$\forall s,t \in [a,b];\ E\left[X_t - X_s\right]^2 \leq M\ \|t-s\|^{2(\beta\wedge H(t))} \tag{19}$$

### 4.1.2 Non-isotropic case

**Lemma 2** *There exists positive constants $K$ and $L$ such that*

$$\forall s,t \in [a,b];\ E\left[|X_t - X_s|^2\right] \leq K\ \|t-s\|^{2\min_i H_i(t)} + L\ \|H(t)-H(s)\|^2 \tag{20}$$

**Proof** By remark 3, we have

$$\begin{aligned} E\left[X_s - X_t\right]^2 &= E\left[\prod_{i=1}^{N} X^{(i)}_{s^{(i)}} - \prod_{i=1}^{N} X^{(i)}_{t^{(i)}}\right]^2 \\ &= E\left[\left(\prod_{i=1}^{N} X^{(i)}_{s^{(i)}} - X^{(1)}_{t^{(1)}}\prod_{i>1} X^{(i)}_{s^{(i)}}\right) + \left(X^{(1)}_{t^{(1)}}\prod_{i>1} X^{(i)}_{s^{(i)}} - X^{(1)}_{t^{(1)}}X^{(2)}_{t^{(2)}}\prod_{i>2} X^{(i)}_{s^{(i)}}\right)\right. \\ &\quad \left. + \cdots + \left(\left(\prod_{i=1}^{N-1} X^{(i)}_{t^{(i)}}\right) X^{(N)}_{s^{(N)}} - \prod_{i=1}^{N} X^{(i)}_{t^{(i)}}\right)\right]^2 \end{aligned}$$

By the inequality of convexity $\left(\sum a_i\right)^2 \leq n \sum a_i^2$, we get

$$E\left[X_s - X_t\right]^2 \leq N\left\{E\left[\prod_{i>1} X^{(i)}_{s^{(i)}}\right]^2 E\left[X^{(1)}_{s^{(1)}} - X^{(1)}_{t^{(1)}}\right]^2 + \cdots + E\left[X^{(N)}_{s^{(N)}} - X^{(N)}_{t^{(N)}}\right]^2 E\left[\prod_{i=1}^{N-1} X^{(i)}_{s^{(i)}}\right]^2\right\}$$

Since there exists a constant $M = M_{a,b}$ such that

$$\forall t \in [a,b], \forall i;\ E\left[X^{(i)}_{t^{(i)}}\right]^2 \leq M$$

we get

$$E[X_s - X_t]^2 \leq NM^{n-1} \sum_{i=1}^{N} E\left[X^{(i)}_{s^{(i)}} - X^{(i)}_{t^{(i)}}\right]^2 \tag{21}$$

Using

$$E\left[X^{(i)}_{s^{(i)}} - X^{(i)}_{t^{(i)}}\right]^2 \leq K_i |s^{(i)} - t^{(i)}|^{2H_i(t)} + L_i \left(H_i(s) - H_i(t)\right)^2 ;\ \forall i = 1, \ldots, N$$

(21) implies

$$E[X_t - X_s]^2 \leq NM^{n-1} \left[\left(\sum_{i=1}^{N} K_i\right) \|t-s\|^{2\min_i H_i(t)} + \left(\sum_{i=1}^{N} L_i\right) \|H(t) - H(s)\|^2\right]$$

□

**Corollary 4** *Suppose $H$ is $\beta$-Hölder continuous. There exists a positive constant $M$ such that*

$$\forall s, t \in [a, b];\ E[X_t - X_s]^2 \leq M \|t-s\|^{2(\beta \wedge \min_i H_i(t))} \tag{22}$$

### 4.1.3 Existence of a continuous modification

In both isotropic and anisotropic cases, under Hölder regularity assumptions for $H$, we have an inequality

$$E[X_t - X_s]^2 \leq K \|t-s\|^{\alpha}$$

But to use the Kolmogorov criterion, we need to have $\alpha > N$.
As the random variable $X_t - X_s$ is Gaussian, we can write, for each integer $n$

$$E[X_t - X_s]^{2n} \leq \lambda_n K \|t-s\|^{n.\alpha}$$

and choose $n$ such that $n.\alpha > N$.
We conclude by a classical patching argument. For $a$ and $b$, Kolmogorov's theorem gives a continuous process $Y^{a,b} = \left\{Y^{a,b}_t ; t \in [a, b]\right\}$. Consider $a'$ and $b'$ such that $[a, b] \subset [a', b']$. The processes $Y^{a,b}$ and $Y^{a',b'}$ coincide on $[a, b]$. Thus

$$\forall t \in [a, b];\ P\left\{Y^{a,b}_t = Y^{a',b'}_t\right\} = 1$$

and, by continuity

$$P\left\{Y^{a,b}_t = Y^{a',b'}_t ; \forall t \in [a, b]\right\} = 1$$

Then we can define a process $Y$ on $\mathbf{R}^N_+$ who coincides with $Y^{a,b}$ on $[a, b]$ and we can seen easily that this process is continuous.

## 4.2 Hölder exponents

The notion of Hölder function is well known. It is interesting to consider a localized version of this notion.

For the paths of a process $X$, one usually define two kinds of exponent (see [1], [2]):

- the pointwise Hölder exponent

$$\begin{aligned}\alpha(t_0) &= \sup\left\{\alpha;\ \lim_{h\to 0}\frac{|X_{t_0+h}-X_{t_0}|}{\|h\|^\alpha}=0\right\}\\ &= \sup\left\{\alpha;\ \limsup_{\rho\to 0}\frac{\sup_{s,t\in B(t_0,\rho)}|X_t-X_s|}{\rho^\alpha}<\infty\right\}\end{aligned}$$

- the local Hölder exponent

$$\tilde{\alpha}(t_0)=\sup\left\{\alpha;\ \limsup_{\rho\to 0}\sup_{s,t\in B(t_0,\rho)}\frac{|X_t-X_s|}{\|t-s\|^\alpha}<\infty\right\}$$

We can see easily that for all $t_0$, we have

$$\tilde{\alpha}(t_0)\leq\alpha(t_0) \tag{23}$$

A study of these exponents, in the case of 1D mBm, is made in [2].

**Remark 4** *If $H$ is $\beta$-Hölder continuous, then the local Hölder exponent $\tilde{\beta}(t)$ of $H$ at every point is not smaller than $\beta$.*

*Conversely, suppose that the local Hölder exponent of $H$ at every point of a compact $[a,b]$ is positive. Then $H$ is $\beta$-Hölder continuous on $[a,b]$ with $\beta=\inf_{t\in[a,b]}\tilde{\beta}(t)$.*

In the following, we suppose that $H$ admits positive local Hölder exponent $\tilde{\beta}(t_0)$ at every point $t$.

**Proposition 13** *Let $X=\left\{X_t;t\in\boldsymbol{R}_+^N\right\}$ be a multifractional Brownian field. For all $t_0\in\boldsymbol{R}_+^N$, the local Hölder exponent of $X$ at $t_0$ is almost surely given by*

$$\tilde{\alpha}(t_0)=\tilde{\beta}(t_0)\wedge H(t_0) \tag{24}$$

*and the pointwise Hölder exponent of $X$ at $t_0$ satisfies almost surely*

$$\alpha(t_0)=\beta(t_0)\wedge H(t_0) \tag{25}$$

*where $\beta(t_0)$ and $\tilde{\beta}(t_0)$ denote the pointwise and local Hölder exponents of $H$ at $t_0$.*

As a consequence of this result, if $H$ satisfies

$$\forall t\in\mathbf{R}_+^N;\ \beta(t)<H(t)$$

the Hölder regularity of multifractional Brownian field of parameter function $H$ is given by the regularity of $H$ (and not by the value of $H$). This point is developed in [7].

**Proposition 14** *Let* $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ *be a multifractional Brownian sheet. For all* $t_0 \in \boldsymbol{R}_+^N$*, the local Hölder exponent of* $X$ *at* $t_0$ *is almost surely given by*

$$\tilde{\alpha}(t_0) = \tilde{\beta}(t_0) \wedge \min_i H_i(t_0) \tag{26}$$

*and the pointwise Hölder exponent of* $X$ *at* $t_0$ *satisfies almost surely*

$$\alpha(t_0) = \beta(t_0) \wedge \min_i H_i(t_0) \tag{27}$$

*where* $\beta(t_0)$ *and* $\tilde{\beta}(t_0)$ *denote the pointwise and local Hölder exponents of* $H$ *at* $t_0$*.*

The proofs of propositions 13 and 14 are detailed in the three following paragraphs.

### 4.2.1 Lower bound for the local Hölder exponent

A lower bound for the local Hölder exponent is directly given by Kolmogorov's theorem. Indeed, for $X$ a multifractional Brownian field or a multifractional Brownian sheet indexed by $[a, b]$, for all $n \in \mathbf{N}$, there exists $\lambda_n > 0$ such that

$$E\left[X_t - X_s\right]^{2n} \leq \lambda_n \|t - s\|^{n.\alpha}$$

with $\alpha = 2\inf_{[a,b]}(\tilde{\beta} \wedge H)$ or $\alpha = 2\inf_{[a,b]}(\tilde{\beta} \wedge \min_i H_i)$.
Kolmogorov's theorem states that there exists a modification of $X$, which is $q$-Hölder continuous for all $q \in (0, \frac{\alpha}{2} - \frac{N}{2n})$. Then, for all $t_0 \in \mathbf{R}_+^N$ and all $a, b \in \mathbf{R}_+^N$ such that $a \prec b$ and $t_0 \in (a, b)$, we have

$$\forall n \in \mathbf{N};\ \tilde{\alpha}(t_0) \geq \frac{\alpha}{2} - \frac{N}{2n}$$

and therefore, taking the limit $n \to \infty$

$$\tilde{\alpha}(t_0) \geq \frac{\alpha}{2}$$

As $H$ is continuous, we can take the limit $(a, b) \to (t_0, t_0)$ and we get

- in the isotropic case,
$$\tilde{\alpha}(t_0) \geq \tilde{\beta}(t_0) \wedge H(t_0) \tag{28}$$
- in the non-isotropic case,
$$\tilde{\alpha}(t_0) \geq \tilde{\beta}(t_0) \wedge \min_i H_i(t_0) \tag{29}$$

### 4.2.2 Lower bound for the pointwise Hölder exponent

By (23), paragraph 4.2.1 provides a lower bound for the pointwise Hölder exponent. However, it can be improved in the case $\tilde{\beta}(t_0) < \beta(t_0)$.
Let $X = \left\{X_t; t \in \mathbf{R}_+^N\right\}$ be a multifractional Brownian field. By corollary 2, there exist positive constants $K$ and $L$ such that for all $s, t \in \mathbf{R}_+^N$,

$$E\left[X_t - X_s\right]^2 \leq K\ \|t - s\|^{2H(t)} + L\ |H(t) - H(s)|^2$$

and by corollary 3, there exists positive constants $\alpha$ and $M$ such that

$$\forall s,t \in [a,b];\ E\left[X_t - X_s\right]^2 \leq M\|t-s\|^\alpha$$

Therefore, using Kolmogorov's criterion, there exists a modification of $X$, which is $\nu$-Hölder continuous for all $\nu \in \left]0, \frac{\alpha}{2}\right[$. In the following, we consider such a $\nu$ with $\frac{1}{\nu} \in \mathbf{N}$
For all $\epsilon > 0$, there exist $\rho_0 > 0$ and $M > 0$ such that for all $\rho < \rho_0$ and all $t \in B(t_0, \rho)$

$$E\left[\frac{X_t - X_{t_0}}{\rho^{\beta(t_0)\wedge H(t_0)-\epsilon}}\right]^2 \leq M\ \rho^\epsilon$$

Then, setting $\gamma = \beta(t_0) \wedge H(t_0) - \epsilon$, for all $p \in \mathbf{N}^*$

$$P\left\{|X_t - X_{t_0}| > \rho^\gamma\right\} \leq E\left[\frac{X_t - X_{t_0}}{\rho^\gamma}\right]^{2p} \leq M_p\ \rho^{p\epsilon}$$

Let $\rho = 2^{-n}$ and for all $m \in \mathbf{N}$,

$$D_m = \left\{t_0 + k.2^{-(n+m)};\ k \in \{0, \pm 1, \ldots, \pm 2^m\}^N\right\}$$

let us compute

$$\begin{aligned} & P\left\{\max_{k\in\{\pm1,\ldots,\pm2^m\}^N} \frac{|X_{t_0+k.2^{-(m+n)}} - X_{t_0}|}{2^{-\gamma n}} > 1\right\} \\ \leq & \sum_{k\in\{\pm1,\ldots,\pm2^m\}^N} P\left\{|X_{t_0+k.2^{-(n+m)}} - X_{t_0}| > 2^{-\gamma n}\right\} \\ \leq & M_p\ 2^{(m+1)N}\ 2^{-p\epsilon n} \end{aligned}$$

Let us take $m = \frac{1+\lfloor\gamma\rfloor}{\nu} n = \kappa n$ and $p \in \mathbf{N}$ such that $N\frac{1+\lfloor\gamma\rfloor}{\nu} - p\epsilon < 0$. By the Borel-Cantelli lemma, there exists a finite random variable $n^*$ such that almost surely,

$$\forall n \geq n^*;\ \max_{k\in\{0,\ldots,\pm2^{\kappa n}\}^N} |X_{t_0+k.2^{-(1+\kappa)n}} - X_{t_0}| \leq 2^{-\gamma n} \tag{30}$$

From (30), we show that, almost surely, for all $m \in \mathbf{N}$, we have

$$\forall t \in D_m;\ |X_t - X_{t_0}| \leq C\ 2^{-\gamma n} \tag{31}$$

- if $0 \leq m \leq \kappa n$, (31) follows directly from (30)
- if $m > \kappa n$, for $t \in D_m$, let

  $$C_{t_0,t}^{\kappa n} = \{x \in D_{\kappa n}; \forall i, (t_0)_i \leq x_i \leq t_i\}$$

  Then consider $\hat{t} \in B(t, 2^{-(1+\kappa)n}) \cap C_{t_0,t}^{\kappa n}$.
  As the paths of $X$ are $\nu$-Hölder continuous, we have

  $$|X_{\hat{t}} - X_t| \leq \tilde{C} 2^{-\nu(1+\kappa)n} \leq \tilde{C} 2^{-\gamma n}$$

  and by (30),

  $$|X_{\hat{t}} - X_{t_0}| \leq 2^{-\gamma n}$$

  Using the triangular inequality, the result follows.

Therefore, (31) leads to

$$\forall m \in \mathbf{N}; \forall s,t \in D_m;\ |X_t - X_s| \leq 2C\ 2^{-\gamma n}$$

Using the continuity of $X$ and $m \to +\infty$, we get

$$\sup_{s,t \in B(t_0, 2^{-n})} |X_t - X_s| \leq 2C\ 2^{-\gamma n}$$

and therefore, almost surely,

$$\limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\rho^\gamma} < +\infty \tag{32}$$

By (32), for all $\epsilon > 0$, almost surely

$$\alpha(t_0) \geq \beta(t_0) \wedge H(t_0) - \epsilon$$

Taking $\epsilon \in \mathbf{Q}_+$, we have almost surely

$$\alpha(t_0) \geq \beta(t_0) \wedge H(t_0) \tag{33}$$

For a multifractional Brownian sheet $X$, by lemma 2, we get in the same way that, almost surely

$$\alpha(t_0) \geq \beta(t_0) \wedge H_i(t_0) \tag{34}$$

for all $i = 1, \ldots, N$.

#### 4.2.3 Upper bound for the pointwise Hölder exponent

The main result getting the upper bound for the Hölder exponents, is the following lemma, a direct consequence of proposition 12 using continuity of $D$, $D'$ and $D''$.

**Lemma 3** *Let $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ be a multifractional Brownian field. For all $[a,b] \subset \boldsymbol{R}_+^N$, there exist positive constants $k_1$, $k_2$, $l_1$, $l_2$ such that*

$$\forall s,t \in [a,b]; \qquad E\left[X_t - X_s\right]^2 \geq k_1\ \|t-s\|^{2H(t)} - l_1\left(H(t) - H(s)\right)^2 \tag{35}$$

$$E\left[X_t - X_s\right]^2 \geq k_2\left(H(t) - H(s)\right)^2 - l_2\ \|t-s\|^{2H(t)} \tag{36}$$

**Proof** We only have to study the multiplicative factors of $\|t-s\|^{2H(t)}$ and $\left(H(t) - H(s)\right)^2$ in (12)

- Let $k_1 = \inf_{t \in [a,b]} D\left[2H(t)\right]$ and $l_2 = \sup_{t \in [a,b]} D\left[2H(t)\right]$.
  By continuity of $t \mapsto D\left[2H(t)\right]$ on the compact $[a,b]$ and as the function $D$ is positive (lemma 1), for all $t \in [a,b]$

$$0 < k_1 \leq D\left[2H(t)\right] \leq l_2 < +\infty$$

- and let

$$\Phi(t) = \|t\|^{2H(t)} \times \left\{D\left[2H(t)\right]\ln^2\|t\| - 2D'\left[2H(t)\right]\ln\|t\| + D''\left[2H(t)\right]\right\}$$

By lemma 1,

$$\Phi(t) = \int_{\mathbf{R}^N} \underbrace{\frac{1-e^{iu_1}}{\|u\|^{2H(t)+N}} \left(\ln\|t\| - \ln\|u\|\right)^2}_{\geq 0} .du$$

Let $k_2 = \inf_{t\in[a,b]} \Phi(t)$ and $l_1 = \sup_{t\in[a,b]} \Phi(t)$.
As previously, for all $t \in [a,b]$, we have

$$0 < k_2 \leq \Phi(t) \leq l_1 < +\infty$$

□

**Lemma 4** *Let $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ be a multifractional Brownian sheet. For all $[a,b] \subset \boldsymbol{R}_+^N$, there exist positive constants $k_1$, $k_2$, $l_1$, $l_2$ such that*

$$\forall s,t \in [a,b]; \qquad t-s \in \boldsymbol{R}_+.\epsilon_i$$

$$E\left[X_t - X_s\right]^2 \geq k_1 \|t-s\|^{2H_i(t)} - l_1 \left(H_i(t) - H_i(s)\right)^2 \quad (37)$$

$$E\left[X_t - X_s\right]^2 \geq k_2 \left(H_i(t) - H_i(s)\right)^2 - l_2 \|t-s\|^{2H_i(t)} \quad (38)$$

**Proof** For all $s$, $t$ such that $t - s \in \mathbf{R}_+.\epsilon_i$, using lemma 3, we have

$$\begin{aligned} E\left[X_t - X_s\right]^2 &= E\left[X^{(i)}_{t^{(i)}} - X^{(i)}_{s^{(i)}}\right]^2 \prod_{j\neq i} E\left[X^{(j)}_{t^{(j)}}\right]^2 \\ &\geq k_1 \left|t_i - s_i\right|^{2H_i(t)} - l_1 \left(H_i(t) - H_i(s)\right)^2 \end{aligned}$$

and

$$E\left[X_t - X_s\right]^2 \geq k_2 \left(H_i(t) - H_i(s)\right)^2 - l_2 \left|t_i - s_i\right|^{2H_i(t)}$$

□

From this result, the upper bound for the pointwise exponent is a consequence of the following lemma whose proof is the same as the case $N = 1$ (see [1])

**Lemma 5** *Let $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ be a Gaussian process. Assume there exists $\mu \in (0,1)$ such that for all $\epsilon > 0$, there exist a sequence $(h_n)_{n\in\boldsymbol{N}}$ of $\left(\boldsymbol{R}_+^N\right)^*$ converging to 0, and a constant $c > 0$ such that*

$$\forall n \in \boldsymbol{N};\ E\left[X_{t+h_n} - X_t\right]^2 \geq c\|h_n\|^{2\mu+\epsilon}$$

*Then we have almost surely*

$$\alpha(t) \leq \mu$$

Let $X = \left\{X_t; t \in \mathbf{R}_+^N\right\}$ be a multifractional Brownian field (resp. multifractional Brownian sheet). Let $\beta(t_0)$ be the pointwise Hölder exponent of $H$ at $t_0$. We consider the two cases :

- if $H(t_0) < \beta(t_0)$ (resp. $H_i(t_0) < \beta(t_0)$), by definition of $\beta(t_0)$, we have

$$\lim_{h \to 0} \frac{\|H(t_0+h) - H(t_0)\|}{\|h\|^{H(t_0)}} = 0$$

Hence, by (35) (resp. (37)), there exists a positive constant $C$ such that

$$E\left[X_{t_0+h} - X_{t_0}\right]^2 \geq C\|h\|^{2H(t_0)}$$

Then, by lemma 5

$$\alpha(t_0) \leq H(t_0) \text{ (resp. } H_i(t_0) \text{ )} \tag{39}$$

- if $H(t_0) > \beta(t_0)$ (resp.$H_i(t_0) > \beta(t_0)$), we consider $\alpha \in (\beta(t_0); H(t_0))$ (resp. $\alpha \in (\beta(t_0); H_i(t_0))$). There exists a positive constant $C$ and a sequence $(h_n)_{n\in\mathbf{N}}$ converging to 0 such that

$$\forall n \in \mathbf{N};\ \|H(t_0+h_n) - H(t_0)\| > C\|h_n\|^{\alpha}$$

Then, by (36) (resp. (38))

$$\begin{aligned} \forall n \in \mathbf{N};\ E\left[X_{t_0+n} - X_{t_0}\right]^2 &> k_2 C\|h_n\|^{2\alpha} - l_2\|h_n\|^{2H(t_0)} \\ &\geq C'\|h_n\|^{2\alpha} \end{aligned}$$

hence, by lemma 5

$$\alpha \geq \alpha(t_0)$$

and therefore

$$\alpha(t_0) \leq \beta(t_0) \tag{40}$$

We can restate the upper bounds (39) and (40) of the pointwise Hölder exponent of $X$ at $t_0$

$$\alpha(t_0) \leq \beta(t_0) \wedge H(t_0) \text{ (resp. } \beta(t_0) \wedge H_i(t_0) \text{ )} \tag{41}$$

### 4.2.4 Upper bound for the local Hölder exponent

By (23), any upper bound for the pointwise Hölder exponent is an upper bound for the local Hölder exponent. But we can improve on this result in the case $\tilde{\beta}(t_0) < H(t_0)$. We first give an analogous of lemma 5 for the local exponent

**Lemma 6** *Let $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ be a Gaussian process. Assume there exists $\mu \in (0,1)$ such that for all $\epsilon > 0$, there exist two sequences $(h_n)_{n\in\boldsymbol{N}}$ and $(l_n)_{n\in\boldsymbol{N}}$ of $\left(\boldsymbol{R}_+^N\right)^*$ converging to 0, and a constant $c > 0$ such that*

$$\forall n \in \boldsymbol{N};\ E\left[X_{t_0+h_n} - X_{t_0+l_n}\right]^2 \geq c\|h_n - l_n\|^{2\mu+\epsilon}$$

*Then we have almost surely*

$$\tilde{\alpha}(t_0) \leq \mu$$

**Proof** Let $\epsilon > 0$ and consider two sequences $(h_n)_{n\in\mathbf{N}}$ and $(l_n)_{n\in\mathbf{N}}$ as in the statement.
For all $n \in \mathbf{N}$, the law of the random variable $\frac{X_{t_0+h_n} - X_{t_0+l_n}}{\|h_n - l_n\|^{\mu+\epsilon}}$ is $\mathcal{N}(0, \sigma_n^2)$.
From the assumption, we have $\sigma_n \to +\infty$ as $n \to +\infty$.
Then, for all $\lambda > 0$,

$$
\begin{aligned}
P\left\{\frac{\|h_n - l_n\|^{\mu+\epsilon}}{|X_{t_0+h_n} - X_{t_0+l_n}|} < \lambda\right\} &= P\left\{\frac{|X_{t_0+h_n} - X_{t_0+l_n}|}{\|h_n - l_n\|^{\mu+\epsilon}} > \frac{1}{\lambda}\right\} \\
&= \int_{|x|>\frac{1}{\lambda}} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{x^2}{2\sigma_n^2}\right).dx \\
&= \frac{1}{2\pi} \int_{|x|>\frac{1}{\lambda\sigma_n}} \exp\left(-\frac{x^2}{2}\right).dx \overset{n\to+\infty}{\longrightarrow} 1
\end{aligned}
$$

Therefore the sequence $\left(\frac{\|h_n - l_n\|^{\mu+\epsilon}}{|X_{t_0+h_n} - X_{t_0+l_n}|}\right)_{n\in\mathbf{N}}$ converges to 0 in probability. then there exists a subsequence which converges to 0 almost surely. Then we have almost surely $\tilde{\alpha}(t_0) \leq \mu + \epsilon$. Taking $\epsilon \in \mathbf{Q}_+$, the result follows.
□

Let $\alpha \in (\tilde{\beta}(t_0); H(t_0))$ (resp. $\alpha \in (\tilde{\beta}(t_0); H_i(t_0))$). As

$$
\limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|H(t) - H(s)|}{\|t - s\|^\alpha} = +\infty
$$

for all $M > 0$, there exists $\rho_0 > 0$ such that

$$
\forall \rho < \rho_0; \exists s, t \in B(t_0, \rho);\ |H(t) - H(s)| > M\|t - s\|^\alpha
$$

Therefore we can construct two sequences $(h_n)$ and $(l_n)$ converging to 0 such that

$$
\forall n \in \mathbf{N};\ |H(t_0 + h_n) - H(t_0 + l_n)| > M\|h_n - l_n\|^\alpha
$$

By lemma 6, we can deduce

$$
\tilde{\alpha}(t_0) \leq \tilde{\beta}(t_0) \tag{42}
$$

## 4.3 Directional Hölder exponents

One may also define directional pointwise and local Hölder exponents in the direction $u \in \mathcal{U} = \left\{u \in \mathbf{R}^N; \|u\| = 1\right\}$ by

$$
\alpha_u(t_0) = \sup\left\{\alpha; \lim_{\rho\to 0} \frac{|X_{t_0+\rho.u} - X_{t_0}|}{\rho^\alpha} = 0\right\}
$$

and

$$
\tilde{\alpha}_u(t_0) = \sup\left\{\alpha;\ \limsup_{\rho\to 0} \sup_{\substack{s,t\in B(t_0,\rho) \\ s,t\in t_0+\mathbf{R}.u}} \frac{|X_t - X_s|}{\|t - s\|^\alpha} < \infty\right\}
$$

As previously, for all $u \in \mathcal{U}$, we have

$$
\tilde{\alpha}_u(t_0) \leq \alpha_u(t_0) \tag{43}
$$

Moreover, we can see easily that for all $u \in \mathcal{U}$, we have

$$\alpha(t_0) \leq \alpha_u(t_0) \text{ and } \tilde{\alpha}(t_0) \leq \tilde{\alpha}_u(t_0) \tag{44}$$

**Proposition 15** *Let $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ be a multifractional Brownian field. For all $t_0 \in \boldsymbol{R}_+^N$ and all $u \in \mathcal{U}$, the local Hölder exponent of $X$ at $t_0$ in the direction u is almost surely given by*

$$\tilde{\alpha}_u(t_0) = \tilde{\beta}_u(t_0) \wedge H(t_0) \tag{45}$$

*and the pointwise Hölder exponent of $X$ at $t_0$ in the direction u satisfies almost surely*

$$\alpha_u(t_0) = \beta_u(t_0) \wedge H(t_0) \tag{46}$$

*where $\beta_u(t_0)$ and $\tilde{\beta}_u(t_0)$ denote the pointwise and local Hölder exponents of $H$ at $t_0$ in the direction $u$.*

**Proof** Let $t_0 \in \mathbf{R}_+^N$, $u \in \mathcal{U}$ and consider the stochastic process

$$\tilde{X} = \left\{\tilde{X}_\rho = X_{t_0+\rho.u}; \rho > 0\right\}$$

By definition, $\alpha_u(t_0)$ and $\tilde{\alpha}_u(t_0)$ are respectively the pointwise and local Hölder exponents of $\tilde{X}$ at 0.
Let $\tilde{H}(\rho) = H(t_0 + \rho.u)$. We have

$$\begin{aligned}
\frac{1}{2} E\left[\tilde{X}_\rho - \tilde{X}_\eta\right]^2 &= \frac{1}{2} E\left[X_{t_0+\rho.u} - X_{t_0+\eta.u}\right]^2 \\
&= D\left[2\tilde{H}(\rho)\right] |\rho - \eta|^{2\tilde{H}(\rho)} + \frac{\partial^2 \varphi}{\partial x^2}\left(2\tilde{H}(\rho); \|t_0 + \rho.u\|\right) \times \left(\tilde{H}(\rho) - \tilde{H}(\eta)\right)^2 \\
&\quad + o\left(\tilde{H}(\rho) - \tilde{H}(\eta)\right)^2 + o\left(|\rho - \eta|^{2\tilde{H}(\rho)}\right)
\end{aligned}$$

by corollary 1.
Then, using the same method as in proposition 13, the result follows. □

**Proposition 16** *Let $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ be a multifractional Brownian sheet. For all $t_0 \in \boldsymbol{R}_+^N$, the local Hölder exponent of $X$ at $t_0$ in the direction $\epsilon_i$ is almost surely given by*

$$\tilde{\alpha}_{\epsilon_i}(t_0) = \tilde{\beta}_{\epsilon_i}(t_0) \wedge H_i(t_0) \tag{47}$$

*and the pointwise Hölder exponent of $X$ at $t_0$ in the direction u satisfies almost surely*

$$\alpha_{\epsilon_i}(t_0) = \beta_{\epsilon_i}(t_0) \wedge H_i(t_0) \tag{48}$$

*where $\beta_{\epsilon_i}(t_0)$ and $\tilde{\beta}_{\epsilon_i}(t_0)$ denote the pointwise and local Hölder exponents of $H$ at $t_0$ in the direction $\epsilon_i$.*

**Proof** As in the proof of lemma 2, there exists a constant $M > 0$ and a one-parameter mBm $X^{(i)}$ such that

$$E\left[X_{t_0+\rho.\epsilon_i} - X_{t_0+\eta.\epsilon_i}\right]^2 = M.E\left[X^{(i)}_{t_0^{(i)}+\rho.\epsilon_i} - X^{(i)}_{t_0^{(i)}+\eta.\epsilon_i}\right]^2$$

Then, using the same method as in proposition 14, the result follows. □

## 4.4 Application of Dudley's theory

Another way to study the regularity of our processes is to examine the behavior around zero of the modulus of continuity

$$\omega_{X,T}(\delta) = \sup_{s,t\in T;d(s,t)\leq\delta} |X_s - X_t|$$

When the process studied is Gaussian, it is convenient to consider the pseudo-metric

$$d(s,t) = E\left[X_s - X_t\right]^2$$

As usually, we define the ball of radius $r > 0$ about $t \in T$ by

$$\mathcal{B}_d(t,r) = \{s \in T; d(s,t) < r\}$$

and we say that $(T,d)$ is totally bounded if for all $\epsilon > 0$, there exists $t_1,\ldots,t_m \in T$ such that

$$T \subset \bigcup_{i=1}^{m} \mathcal{B}_d(t_i,\epsilon)$$

When $(T,d)$ is totally bounded, we can define the metric entropy $\epsilon \mapsto D(\epsilon,T,d)$ where $D(\epsilon,T,d)$ is minimum number of balls of radius $\epsilon$ required to cover $T$.

The following theorem allows to improve on the results of the previous paragraph.

**Theorem 2 (Dudley's Theorem)** *Consider a centered Gaussian process* $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ *indexed by the pseudo-metric space* $(T,d)$. *If* $(T,d)$ *is totally bounded and if* $\int_0^1 \sqrt{\ln D(r,T,d)}dr < \infty$, *then* $X$ *has a continuous modification* $Y = \left\{Y_t; t \in \boldsymbol{R}_+^N\right\}$. *Moreover, there exists a universal constant* $C > 0$ *such that*

$$\limsup_{\delta\to 0^+} \frac{\omega_{Y,T}(\delta)}{\int_0^\delta \sqrt{\ln D(\frac{r}{2},T,d)}dr + C\delta\sqrt{\ln\ln\frac{1}{\delta}}} \leq 24$$

To apply this result, we first need to verify the assumptions about the metric entropy.

**Lemma 7** *Let* $T \subset \boldsymbol{R}_+^N$ *measurable and* $d$ *a pseudo-metric on* $T$. *If there exists* $C > 0$ *and* $\alpha > 0$ *such that*

$$\forall s,t \in T;\ d(s,t) \leq C\|s-t\|^\alpha$$

*then there exists* $r_0 > 0$ *such that* $\forall r \in [0,r_0]$,

$$D(r,T,d) \leq C^{\frac{N}{\alpha}}.Leb(T).r^{-\frac{N}{\alpha}}$$

We saw previously that the 2 multi-parameter extensions of the mBm, satisfy the assumption of this lemma with $T = [a, b]$. Then there exists $C_{a,b} > 0$, $\alpha = \alpha_{a,b} > 0$ and $r_0 > 0$ such that for all $r \in [0, t_0]$,

$$D(r, [a, b], d) \leq C^{\frac{N}{\alpha}}.Leb([a, b]).r^{-\frac{N}{\alpha}}$$

As a consequence, $([a, b], d)$ is totally bounded and in the neighborhood of 0, we have

$$\sqrt{\ln D(r, T, d)} \leq \sqrt{K - \frac{N}{\alpha} \ln r}$$

therefore the integral $\int_0^1 \sqrt{\ln D(r, T, d)}dr$ is finite and we can apply Dudley's theorem. We get

**Proposition 17** *Let* $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ *be one of the multi-parameter extension of the mBm. For all* $a \prec b$*, there exists* $C_{a,b} > 0$ *and* $\alpha = \alpha_{a,b} > 0$ *such that*

$$\limsup_{\epsilon \to 0} \frac{\sup_{\|s-t\| \leq \epsilon} |X_s - X_t|}{\epsilon^\alpha \sqrt{\ln \frac{1}{\epsilon}}} \leq C_{a,b}\sqrt{N}$$

**Proof** First of all, we study the quotient

$$\frac{\int_0^\delta \sqrt{\ln D(\frac{r}{2}, T, d)}dr + C\delta\sqrt{\ln\ln\frac{1}{\delta}}}{\delta\sqrt{\ln\frac{1}{\delta}}} = \frac{N(\delta)}{D(\delta)}$$

The derivative of the numerator is

$$\begin{aligned} N'(\delta) &= \sqrt{\ln D(\frac{\delta}{2}, T, d)} + C\left[\sqrt{\ln\ln\frac{1}{\delta}} - \frac{1}{2\ln\frac{1}{\delta}\sqrt{\ln\ln\frac{1}{\delta}}}\right] \\ &\leq \sqrt{K - \frac{N}{\alpha}\ln\delta} + C\sqrt{\ln\ln\frac{1}{\delta}}\left[1 - \frac{1}{2\ln\frac{1}{\delta}\ln\ln\frac{1}{\delta}}\right] \end{aligned}$$

and the derivative of the denominator is

$$\begin{aligned} D'(\delta) &= \sqrt{\ln\frac{1}{\delta}}\left[1 - \frac{1}{2\ln\frac{1}{\delta}}\right] \\ &\sim \sqrt{\ln\frac{1}{\delta}} \end{aligned}$$

Then we have

$$\limsup_{\delta \to 0+} \frac{N'(\delta)}{D'(\delta)} \leq \sqrt{\frac{N}{\alpha}}$$

and by a L'Hopital's rule type argument,

$$\limsup_{\delta \to 0+} \frac{N(\delta)}{D(\delta)} \leq \sqrt{\frac{N}{\alpha}}$$

Then we have

$$\limsup_{\delta\to 0} \frac{\sup_{d(s,t)\le\delta} |X_s - X_t|}{\delta\sqrt{\ln\frac{1}{\delta}}} \le 24\sqrt{\frac{N}{\alpha}}$$

The problem is now to transform $\sup_{d(s,t)\le\delta}$ into $\sup_{\|s-t\|\le\delta}$.
To do this, we write

$$\forall s,t \in [a,b];\ d(s,t) \le C_{a,b}\|s-t\|^{\alpha}$$

then $\|s-t\| \le \epsilon = \frac{\delta^{\frac{1}{\alpha}}}{C_{a,b}^{\frac{1}{\alpha}}}$ implies $d(s,t)\le\delta$ and we get

$$\limsup_{\delta\to 0} \frac{\sup_{\|s-t\|\le\epsilon} |X_s - X_t|}{C_{a,b}\epsilon^{\alpha}\sqrt{\alpha\ln\frac{1}{\epsilon}}} \le 24\sqrt{\frac{N}{\alpha}}$$

which gives the expected result. □

This result is more powerful than knowledge of Hölder exponents. It gives the behavior of $|X_s - X_t|$ in a ball around $t_0$.

# 5 Locally asymptotic self-similarity

Extending fBm into multifractional Brownian motion implies the loss of the two properties of self-similarity and stationarity of increments. However, a weak form of self-similarity remains, called locally asymptotic self-similarity (see [1], [4]). As we will see, this property still holds for the two kinds of extension of mBm in $\mathbf{R}^N$.

**Theorem 3** *Let $X = \left\{X_t; t\in \boldsymbol{R}_+^N\right\}$ be a multifractional Brownian field.*
*For all $t_0 \in \boldsymbol{R}_+^N$, the law of the process $Y^{\alpha}(\rho) = \left\{Y_u^{\alpha}(\rho) = \frac{X_{t_0+\rho u} - X_{t_0}}{\rho^{\alpha}}; u \in \boldsymbol{R}_+^N\right\}$ converges weakly if one of the following two conditions holds*

1. *$\alpha = H(t_0)$ and $H(t_0) < \inf_{u,v}\beta_{uv}(t_0)$*
   *where $\beta_{uv}(t_0) = \sup\left\{\alpha;\ \lim_{\rho\to 0}\frac{|H(t_0+\rho u)-H(t_0+\rho v)|}{\rho^{\alpha}} = 0\right\}$.*
   *Then, the limit measure is the law of a fractional Brownian field with parameter $H(t_0)$.*

2. *$\alpha = \inf_{u,v}\beta_{uv}(t_0)$, $H(t_0) > \inf_{u,v}\beta_{uv}(t_0)$ and for all $u,v \in \boldsymbol{R}_+^N$, the following limit exists*

   $$\lim_{\rho\to 0}\frac{|H(t_0+\rho u) - H(t_0+\rho v)|}{\rho^{\inf_{u,v}\beta_{uv}(t_0)}} = \Gamma(u,v)$$

   *with $(u,v)\mapsto \frac{\Gamma(u,v)}{\|u-v\|^{2\beta}}$ bounded on $[a,b]^2$ for some $\beta > 0$.*
   *The limit measure is the law of a Gaussian process $Y^{\inf_{u,v}\beta_{uv}(t_0)}$ such that*

   $$E\left[Y_u^{\inf_{u,v}\beta_{uv}(t_0)} - Y_v^{\inf_{u,v}\beta_{uv}(t_0)}\right]^2 = K_{t_0}\left[\Gamma(u,v)\right]^2$$

**Remark 5** *As in the Levy fBm's case in proposition 6, the same result as theorem 3 can be stated for the increments* $\Delta X$ *defined in section 2.3. The law of the process* $Y^\alpha(\rho) = \left\{Y_u^\alpha(\rho) = \frac{\Delta X_{t_0,t_0+\rho u}}{\rho^\alpha}; u \in \mathbf{R}_+^N\right\}$ *converges weakly under the same assumptions.*

In the case $N = 1$, for all $u, v \in \mathbf{R}_+$, we have $\beta_{uv}(t_0) = \beta(t_0)$. Therefore, theorem 3 has a simpler statement. The two cases to be considered, depend of the comparison between $H(t_0)$ and the pointwise exponent $\beta(t_0)$ of $H$.

The following example shows that the limit considered in the second case, can be non trivial.

**Example 1** *In the case* $N = 1$, *let* $H(t) = \frac{3}{4} + t^{\frac{1}{2}}$ *for* $t \in [0, \frac{1}{4}]$.
*For* $t_0 = 0$, *we compute, for all* $u, v$ *and* $\rho > 0$

$$\frac{|H(\rho.u) - H(\rho.v)|}{\rho^{\frac{1}{2}}} = |u^{\frac{1}{2}} - v^{\frac{1}{2}}| < |u - v|^{\frac{1}{2}}$$

*The limit measure is the law of a centered Gaussian process* $Y$ *such that*

$$E\left[Y_u - Y_v\right]^2 = K_0 \left(u^{\frac{1}{2}} - v^{\frac{1}{2}}\right)^2$$

*ie*

$$E\left[Y_u Y_v\right] = K_0\; u^{\frac{1}{2}} v^{\frac{1}{2}}$$

**Theorem 4** *Let* $X = \left\{X_t; t \in \mathbf{R}_+^N\right\}$ *be a multifractional Brownian sheet.*
*The law of the process* $Y^\alpha(\rho) = \left\{Y_u^\alpha(\rho) = \frac{\Delta X_{t_0,t_0+\rho u}}{\rho^{\sum_i \alpha_i}}; u \in \mathbf{R}_+^N\right\}$ *converges weakly if for all* $i \in \{1, \ldots, N\}$, *one of the following two conditions holds*

1. $\alpha_i = H_i(t_0)$ *and* $H_i(t_0) < \inf_{u,v} \beta_{uv}^i(t_0)$
   *where* $\beta_{uv}^i(t_0) = \sup\left\{\alpha;\ \lim_{\rho\to 0} \frac{|H_i(t_0+\rho u) - H_i(t_0+\rho v)|}{\rho^\alpha} = 0\right\}$.

2. $\alpha_i = \inf_{u,v} \beta_{uv}^i(t_0)$, $H_i(t_0) > \inf_{u,v} \beta_{uv}^i(t_0)$ *and*

$$\lim_{\rho\to 0} \frac{|H_i(t_0 + \rho u) - H_i(t_0 + \rho v)|}{\rho^{\inf_{u,v} \beta_{uv}^i(t_0)}} = \Gamma_i(u, v)$$

   *with* $(u, v) \mapsto \frac{\Gamma_i(u,v)}{\|u-v\|^{2\beta_i}}$ *bounded on* $[a, b]^2$ *for some* $\beta_i > 0$.

As usually, the proof of weak convergence proceeds in two steps. First, we need to show finite dimensional convergence, and then, use a tightness argument. Lemma 14.2 and theorem 14.3 in [10], for instance, allow then to conclude.

## 5.1 Finite dimensional convergence

As the considered processes are Gaussian, we only have to show the convergence of covariance functions.

### 5.1.1 Multifractional Brownian field

By (12), we compute

$$\begin{aligned}\rho^{2\alpha}E\left[Y_u^\alpha(\rho)-Y_v^\alpha(\rho)\right]^2 &= E\left[X_{t_0+\rho u}-X_{t_0+\rho v}\right]^2\\ &= D\left[H(t_0+\rho u)+H(t_0+\rho v)\right]\times\|\rho.(u-v)\|^{H(t_0+\rho u)+H(t_0+\rho v)}\\ &\quad +\frac{\partial^2\varphi}{\partial x^2}\left(2H(t_0+\rho u);\|t_0+\rho u\|\right)\times\left(H(t_0+\rho u)-H(t_0+\rho v)\right)^2\\ &\quad +o\left(\|\rho.(u-v)\|^2\right)+o\left(H(t_0+\rho u)-H(t_0+\rho v)\right)^2 \qquad (49)\end{aligned}$$

To show that

$$\rho^{H(t_0+\rho u)+H(t_0+\rho v)}\sim\rho^{2H(t_0)}$$

in the neighborhood of $\rho=0$, we study

$$\begin{aligned}\left[H(t_0+\rho u)+H(t_0+\rho v)-2H(t_0)\right]\,\ln\rho &= \frac{H(t_0+\rho u)-H(t_0)}{\|\rho.u\|^\alpha}\times\|\rho.u\|^\alpha\ln\rho\\ &\quad+\frac{H(t_0+\rho v)-H(t_0)}{\|\rho.v\|^\alpha}\times\|\rho.v\|^\alpha\ln\rho\end{aligned}$$

for $\alpha<\beta(t_0)$.
As $(u;\rho)\mapsto\|\rho.u\|^\alpha\ln\rho$ is bounded on $[a,b]\times[0,1]$ and

$$\forall u\in[a,b];\ \frac{H(t_0+\rho u)-H(t_0)}{\|\rho.u\|^\alpha}\overset{\rho\to 0}{\longrightarrow}0$$

we have

$$\left[H(t_0+\rho u)+H(t_0+\rho v)-2H(t_0)\right]\,\ln\rho\overset{\rho\to 0}{\longrightarrow}0$$

Therefore, in the neighborhood of $\rho=0$, the first term of (49) is equivalent to

$$D\left[2H(t_0)\right]\|u-v\|^{2H(t_0)}\times\rho^{2H(t_0)}$$

and the second to

$$\frac{\partial^2\varphi}{\partial x^2}\left(2H(t_0);\|t_0\|\right)\times\left(H(t_0+\rho u)-H(t_0+\rho v)\right)^2$$

Let $\beta_{uv}(t_0)=\sup\left\{\alpha;\lim_{\rho\to 0}\frac{|H(t_0+\rho u)-H(t_0+\rho v)|}{\rho^\alpha}=0\right\}$. We have to distinguish the two following cases

- if $H(t_0)<\inf_{u,v}\beta_{uv}(t_0)$, by definition of $\beta_{uv}(t_0)$,

  $$\forall u,v\in\mathbf{R}_+^N;\ \lim_{\rho\to 0}\frac{|H(t_0+\rho u)-H(t_0+\rho v)|}{\rho^{H(t_0)}}=0$$

  Therefore

  $$\forall u,v\in\mathbf{R}_+^N;\ E\left[Y_u^{H(t_0)}(\rho)-Y_v^{H(t_0)}(\rho)\right]^2\overset{\rho\to 0}{\longrightarrow}\underbrace{D\left[2H(t_0)\right]\|u-v\|^{2H(t_0)}}_{E\left[B_u^{H(t_0)}-B_v^{H(t_0)}\right]^2}$$

  where $B^{H(t_0)}$ denotes fractional Brownian field of parameter $H(t_0)$.

- if $H(t_0) > \inf_{u,v} \beta_{uv}(t_0)$,
  for all $\alpha < \inf_{u,v} \beta_{uv}(t_0)$, as

$$\forall u, v \in \mathbf{R}_+^N;\ \lim_{\rho \to 0} \frac{|H(t_0 + \rho u) - H(t_0 + \rho v)|}{\rho^\alpha} = 0$$

we have

$$\forall u, v \in \mathbf{R}_+^N;\ \frac{1}{\rho^{2\alpha}} E\left[X_{t_0+\rho u} - X_{t_0+\rho v}\right]^2 \stackrel{\rho\to 0}{\longrightarrow} 0$$

Moreover, since there exists $u, v \in \mathbf{R}_+^N$ such that $H(t_0) > \beta_{uv}(t_0)$, we can consider $\alpha \in (\beta_{uv}(t_0); H(t_0))$. The limit

$$\limsup_{\rho \to 0} \frac{|H(t_0 + \rho u) - H(t_0 + \rho v)|}{\rho^\alpha} = +\infty$$

implies

$$\exists u, v \in \mathbf{R}_+^N;\ \limsup_{\rho \to 0} \frac{1}{\rho^{2\alpha}} E\left[X_{t_0+\rho u} - X_{t_0+\rho v}\right]^2 = +\infty$$

Therefore $E\left[Y_u^\alpha(\rho) - Y_v^\alpha(\rho)\right]^2$ admits a limit for all $u, v \in \mathbf{R}_+^N$ when $\rho \to 0$ **if and only if**

$$\begin{cases} \alpha = \inf_{u,v} \beta_{uv}(t_0) \\ \text{and} \\ \lim_{\rho\to 0} \frac{|H(t_0+\rho u) - H(t_0+\rho v)|}{\rho^{\inf_{u,v} \beta_{uv}(t_0)}} = \Gamma(u,v) \in \mathbf{R}_+^* \end{cases}$$

In that case, we have for all $u$, $v$ in $\mathbf{R}_+^N$,

$$E\left[Y_u^\alpha(\rho) - Y_v^\alpha(\rho)\right]^2 \stackrel{\rho\to 0}{\longrightarrow} \frac{\partial^2 \varphi}{\partial x^2}\left(2H(t_0); \|t_0\|\right) \left[\Gamma(u,v)\right]^2$$

**Remark 6** *We can see easily that*

$$\beta_{\frac{u}{\|u\|}}(t_0) \wedge \beta_{\frac{v}{\|v\|}}(t_0) \le \beta_{uv}(t_0) \tag{50}$$

*hence*

$$\inf_{u \in \mathcal{U}} \beta_u(t_0) \le \inf_{u,v} \beta_{uv}(t_0) \tag{51}$$

*Conversely, assume there exist $u, v \in \mathcal{U}$ such that $\beta_u(t_0) < \beta_v(t_0)$, and let $\alpha \in (\beta_u(t_0); \beta_v(t_0))$. The inequality*

$$\frac{|H(t_0 + \rho u) - H(t_0)|}{\rho^\alpha} \le \frac{|H(t_0 + \rho u) - H(t_0 + \rho v)|}{\rho^\alpha} + \frac{|H(t_0 + \rho v) - H(t_0)|}{\rho^\alpha}$$

*implies*

$$\limsup_{\rho \to 0} \frac{|H(t_0 + \rho u) - H(t_0 + \rho v)|}{\rho^\alpha} = +\infty$$

*and therefore $\alpha > \beta_{uv}(t_0)$. Then $\inf_{u,v} \beta_{uv}(t_0) \le \inf_{u\in\mathcal{U}} \beta_u(t_0)$, which gives*

$$\inf_{u,v} \beta_{uv}(t_0) = \inf_{u \in \mathcal{U}} \beta_u(t_0) \tag{52}$$

### 5.1.2 Multifractional Brownian sheet

In the non-isotropic case, using remark 3, consider $N$ independent mBm $X^{(i)}$ with parameter function $H_i$

$$E\left[\Delta X_{t_0,t_0+\rho u}\Delta X_{t_0,t_0+\rho v}\right]=\prod_{i=1}^{N}E\left[\left(X^{(i)}_{t_0^{(i)}+\rho u^{(i)}}-X^{(i)}_{t_0^{(i)}}\right)\left(X^{(i)}_{t_0^{(i)}+\rho v^{(i)}}-X^{(i)}_{t_0^{(i)}}\right)\right]$$

As in the isotropic case, for all $i\in\{1,\dots,N\}$, consider

$$\beta^i_{uv}(t_0)=\sup\left\{\alpha;\ \lim_{\rho\to 0}\frac{|H_i(t_0+\rho u)-H_i(t_0+\rho v)|}{\rho^\alpha}=0\right\}$$

Each process $X^{(i)}$ is locally asymptoticly self-similar, therefore

$$E\left[\frac{X^{(i)}_{t_0^{(i)}+\rho u^{(i)}}-X^{(i)}_{t_0^{(i)}}}{\rho^{\alpha_i}}\times\frac{X^{(i)}_{t_0^{(i)}+\rho v^{(i)}}-X^{(i)}_{t_0^{(i)}}}{\rho^{\alpha_i}}\right]\stackrel{\rho\to 0}{\longrightarrow}E\left[Y^{\alpha_i}_uY^{\alpha_i}_v\right]$$

where $Y^{\alpha_i}$ denotes

- fractional Brownian motion of parameter $\alpha_i=H_i(t_0)$, in the case $H_i(t_0)<\inf_{u,v}\beta^i_{uv}(t_0)$,
- the centered Gaussian process such that

  $$E\left[Y^{\alpha_i}_u-Y^{\alpha_i}_v\right]^2=K_{t_0}\left[\Gamma_i(u,v)\right]^2$$

  where $\alpha_i=\inf_{u,v}\beta^i_{uv}(t_0)$, in the case $H_i(t_0)>\inf_{u,v}\beta^i_{uv}(t_0)$ and

  $$\lim_{\rho\to 0}\frac{|H_i(t_0+\rho u)-H_i(t_0+\rho v)|}{\rho^{\inf_{u,v}\beta^i_{uv}(t_0)}}=\Gamma_i(u,v)$$

  with $\Gamma_i$ bounded on $[a,b]^2$.

Then we conclude

$$E\left[\frac{\Delta X^{(i)}_{t_0^{(i)},t_0^{(i)}+\rho u^{(i)}}}{\rho^{\sum_i\alpha_i}}\times\frac{\Delta X^{(i)}_{t_0^{(i)},t_0^{(i)}+\rho v^{(i)}}}{\rho^{\sum_i\alpha_i}}\right]\stackrel{\rho\to 0}{\longrightarrow}E\left[Y^\alpha_uY^\alpha_v\right]$$

where $Y^\alpha=\prod_{i=1}^N Y^{\alpha_i}$.

## 5.2 Tightness of laws

The study of weak convergence is well-known for stochastic processes indexed by $\mathbf{R}_+$. A comprehensive review was made by Billingsley (cf [5]) for a compact set of index ([0, 1]). In ([11]), Karatzas and Shreeve stated the same kind of results for the whole $\mathbf{R}_+$. The case of $\mathbf{R}_+^N$ can be found in ([10]) whose corollary 14.9 provides

**Proposition 18** *Consider a sequence of continuous processes* $\left(X^{(n)}\right)_{n\in\boldsymbol{N}}$ *with* $X^{(n)}=\left\{X^{(n)}_t;t\in\boldsymbol{R}_+^N\right\}$ *on* $(\Omega,\mathcal{F},P)$ *such that*

1. *there exists a positive constant* $\nu$ *such that*

$$\sup_{n\geq 1} E\left|X_0^{(n)}\right|^{\nu} < \infty$$

2. *for all* $T > 0$ *and all* $s$, $t$ *in* $[0,T]^N$, *there exist positive constants* $\alpha$, $\beta$ *and* $C_T$ *such that*

$$\sup_{n\geq 1} E\left|X_t^{(n)} - X_s^{(n)}\right|^{\alpha} \leq C_T \|t-s\|^{N+\beta}$$

*Then the probability measures* $P_n \triangleq P.\left(X^{(n)}\right)^{-1}$ *on* $\left(C\left(\boldsymbol{R}_+^N\right), \mathcal{B}\left(C\left(\boldsymbol{R}_+^N\right)\right)\right)$ *form a tight sequence.*

We verify the conditions of proposition 18, in the case of mBm, in the following sections.

### 5.2.1 Multifractional Brownian field

By (18), there exist positive constants $K_T$ and $L_T$ such that for all $u$, $v$ in $[0,T]^N$

$$\begin{aligned}
\rho^{2\alpha} E\left[Y_u^{\alpha}(\rho) - Y_v^{\alpha}(\rho)\right]^2 &= E\left[X_{t_0+\rho u} - X_{t_0+\rho v}\right]^2 \\
&\leq K_T \|\rho.(u-v)\|^{2H(t_0+\rho u)} \\
&\quad + L_T \left|H(t_0+\rho u) - H(t_0+\rho v)\right|^2
\end{aligned}$$

Therefore,

$$E\left[Y_u^{\alpha}(\rho) - Y_v^{\alpha}(\rho)\right]^2 \leq K_T' \,\rho^{2(H(t_0)-\alpha)}.\|(u-v)\|^{2H(t_0)} + L_T \frac{\left|H(t_0+\rho u) - H(t_0+\rho v)\right|^2}{\rho^{2\alpha}}$$

- In the case $H(t_0) < \inf_{u,v} \beta_{uv}(t_0)$, there exists $M_T > 0$ such that

$$E\left[Y_u^{H(t_0)}(\rho) - Y_v^{H(t_0)}(\rho)\right]^2 \leq M_T \|u-v\|^{2H(t_0)}$$

- In the case $H(t_0) > \inf_{u,v} \beta_{uv}(t_0)$, under the assumption

$$\lim_{\rho\to 0} \frac{\left|H(t_0+\rho u) - H(t_0+\rho v)\right|}{\rho^{\inf_{u,v}\beta_{uv}(t_0)}} = \Gamma(u,v)$$

with $(u,v) \mapsto \frac{\Gamma(u,v)}{\|u-v\|^{2\beta}}$ bounded on $[a,b]^2$, there exists $M_T > 0$ such that

$$E\left[Y_u^{\inf_{u,v}\beta_{uv}(t_0)}(\rho) - Y_v^{\inf_{u,v}\beta_{uv}(t_0)}(\rho)\right]^2 \leq M_T \|u-v\|^{2(\beta\wedge H(t_0))}$$

Since the process $Y^{\alpha}$ is Gaussian, we get an exponent greater than $N$ in the usual way. Then we can conclude by proposition 18 that the laws of $Y^{\alpha}$ are tight.

### 5.2.2 Multifractional Brownian sheet

In the same way as in paragraph 4.1.2

$$\begin{aligned} E\left[Y_u^{(\rho)} - Y_v^{(\rho)}\right]^2 &= \frac{1}{\rho^2 \sum_i \alpha_i} E\left[\prod_{i=1}^N \left(X^{(i)}_{t_0^{(i)}+\rho u^{(i)}} - X^{(i)}_{t_0^{(i)}}\right) - \prod_{i=1}^N \left(X^{(i)}_{t_0^{(i)}+\rho v^{(i)}} - X^{(i)}_{t_0^{(i)}}\right)\right] \\ &\leq K \sum_i E\left[\frac{X^{(i)}_{t_0^{(i)}+\rho u^{(i)}} - X^{(i)}_{t_0^{(i)}+\rho v^{(i)}}}{\rho^{\alpha_i}}\right]^2 \end{aligned}$$

then, under the assumptions of theorem 4, there exists a positive constant $M_T$ such that

$$E\left[Y_u^{(\rho)} - Y_v^{(\rho)}\right]^2 \leq M_T \, \|u - v\|^{2 \min_i \alpha_i}$$

We conclude as in the isotropic case.

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# Chapitre III

## A set-indexed fractional Brownian motion

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# A set-indexed fractional Brownian motion

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**Abstract**

We define and prove the existence of a fractional Brownian motion indexed by a collection of closed subsets of a measure space. This process is a generalization of the set-indexed Brownian motion, when the condition of independance is relaxed. Relations with the Lévy fractional Brownian motion and with the fractional Brownian sheet are studied. We prove stationarity of the increments and a property of self-similarity with respect to the action of solid motions. Regularity conditions are exhibited. Finally, behavior of the set-indexed fractional Brownian motion along increasing paths is analysed.

*AMS classification*: 62 G 05, 60 G 15, 60 G 17, 60 G 18.
*Keywords*: fractional Brownian motion, Gaussian processes, stationarity, self-similarity, set-indexed processes.

## 1 Introduction

Recently, different extensions of the fractional Brownian motion were introduced and widely used to describe complex or chaotic phenomena in several fields of sciences. Here we define a new very natural set-indexed generalization of the fractional Brownian motion. The set-indexed fractional Brownian motion studied here seems to be well-adapted to modelize problems in applied mathematics (see [11]).

The fractional Brownian motion was defined by B. B. Mandelbrot and J. W. Van Ness, and extended essentially into two directions. One is generally called the multifractional Brownian motion, replacing the index parameter of self-similarity (called also the Hurst parameter) by a real measurable function (see [2], [10]). The other one are multiparameter fractional Brownian motions in which the set of the indices is a subset of the Euclidean space $\mathbf{R}^N$ (see [8] and [12], [14] for trajectory properties).

At least two types of multiparameter fractional Brownian motions were introduced. One is called the Lévy fractional Brownian motion because it extends

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the Lévy Brownian motion and the other is called the fractional Brownian sheet because it can be seen as an extension of the Brownian sheet. We refer to [4] for a survey on all these processes. Moreover, in [4], some multiparameter extensions of the multifractional Brownian motion are well studied.

The set-indexed fractional Brownian motion introduced here is a simple extension of the set-indexed Brownian motion (called also the white noise) and possesses the main properties required for a fractional Brownian motion. Moreover, by choosing the class of sets parametrizing the process, we get great flexibility and possibilities to obtain particular types of fractional Brownian motion.

In this paper, we prove that our definition of set-indexed fractional Brownian motion satisfies both self-similarity and a condition of stationarity. Moreover such a process is a time-changed classical fractional Brownian motion on each increasing path (flow).
Conversely, we compute the covariance function of any self-similar and stationary set-indexed process. For Gaussian processes, we get an extension of our set-indexed fractional Brownian motion.

This paper is divided as follow :
In the next section, we present the general framework needed for set-indexed processes. We define the concept of set-indexed fractional Brownian motion (SifBm). We prove the existence of this process showing that its covariance function is positive definite. Moreover, we compare our definition to previous definitions given in some particular cases and our definition seems to be quite natural and satisfactory. The two fractal properties which are stationarity and self-similarity are studied in section 3. As it will be see, stationarity of increments can be defined in different non equivalent ways. In section 4, we discuss the possibility of finding a characterization of the set-indexed fractional Brownian motion by the two main properties: stationarity and self-similarity. Section 5 deals with the problem of continuity. In fact, we show that the SifBm is continuous when the set-indexed Brownian motion is also continuous. Finally in the last section, we study SifBm on increasing paths.

# 2 Framework and definition

## 2.1 Indexing collection, set-indexed processes

Let $\mathcal{T}$ be a locally compact complete separable metric and measure space with metric $d$ and measure $m$. All processes will be indexed by a class $\mathcal{A}$ of compact connected subsets of $\mathcal{T}$.

In what follows, for any class of sets $\mathcal{D}$, the class of finite unions of sets from $\mathcal{D}$ will be denoted by $\mathcal{D}(u)$. In the terminology of [7], we assume that $\mathcal{A}$ is an *indexing collection*:

**Definition 2.1** *A nonempty class $\mathcal{A}$ of compact, connected subsets of $\mathcal{T}$ is called an* indexing collection *if it satisfies the following:*

1. $\emptyset$, $\mathcal{T} \in \mathcal{A}$, *and* $A^\circ \neq A$ *if* $A \neq \emptyset$ *or* $\mathcal{T}$.
2. $\mathcal{A}$ *is closed under arbitrary intersections and if* $A, B \in \mathcal{A}$ *are nonempty, then* $A \cap B$ *is nonempty. If* $(A_i)$ *is an increasing sequence in* $\mathcal{A}$ *then* $\overline{\bigcup_i A_i} \in \mathcal{A}$.

3. *The $\sigma$-algebra generated by $\mathcal{A}$, $\sigma(\mathcal{A}) = \mathcal{B}$, the collection of all Borel sets of $\mathcal{T}$.*

4. Separability from above
*There exists an increasing sequence of finite subclasses $\mathcal{A}_n = \{A_1^n, ..., A_{k_n}^n\}$ of $\mathcal{A}$ closed under intersections and satisfying $\emptyset, \mathcal{T} \in \mathcal{A}_n$ and a sequence of functions $g_n : \mathcal{A} \to \mathcal{A}_n(u)$ such that*

   (a) *$g_n$ preserves arbitrary intersections and finite unions (i.e. $g_n(\bigcap_{A \in \mathcal{A}'} A) = \bigcap_{A \in \mathcal{A}'} g_n(A)$ for any $\mathcal{A}' \subseteq \mathcal{A}$, and if $\bigcup_{i=1}^k A_i = \bigcup_{j=1}^m A_j'$, then $\bigcup_{i=1}^k g_n(A_i) = \bigcup_{j=1}^m g_n(A_j')$),*

   (b) *for each $A \in \mathcal{A}$, $A \subseteq (g_n(A))^\circ$,*

   (c) *$g_n(A) \subseteq g_m(A)$ if $n \geq m$,*

   (d) *for each $A \in \mathcal{A}$, $A = \bigcap_n g_n(A)$ ,*

   (e) *if $A, A' \in \mathcal{A}$ then for every $n$, $g_n(A) \cap A' \in \mathcal{A}$, and if $A' \in \mathcal{A}_n$ then $g_n(A) \cap A' \in \mathcal{A}_n$.*

   (f) *$g_n(\emptyset) = \emptyset \ \forall n$.*

5. *Every countable intersection of sets in $\mathcal{A}(u)$ may be expressed as the closure of a countable union of sets in $\mathcal{A}$.*

*(Note: '$\subset$' indicates strict inclusion and '$\overline{(\cdot)}$' and '$(\cdot)^\circ$' denote respectively the closure and the interior of a set.)*

We shall define the semi-algebra $\mathcal{C}$ to be the class of all subsets of $\mathcal{T}$ of the form

$$C = A \setminus B, \ A \in \mathcal{A}, \ B \in \mathcal{A}(u).$$

$\mathcal{C}$ is closed under intersections and any set in $\mathcal{C}(u)$ may be expressed as a finite disjoint union of sets in $\mathcal{C}$. Note that if $B = \bigcup_{i=1}^k A_i \in \mathcal{A}(u)$, without loss of generality we can require that for each $i$, $A_i \not\subseteq \bigcup_{j \neq i} A_j$. Such a representation of $B \in \mathcal{A}(u)$ will be called *extremal.* If $C = A \setminus B$, $A \in \mathcal{A}$, $B \in \mathcal{A}(u)$, then the representation of $C$ is called extremal if that of $B$ is. Unless otherwise stated, it will always be assumed that all representations of sets in $\mathcal{A}(u)$ and $C$ are extremal.

Numerous examples of topological spaces $\mathcal{T}$ and indexing collections $\mathcal{A}$ satisfying the preceding assumptions may be found in [7]. In particular, our framework generalizes the usual multiparameter setting: if $\mathcal{T} = \mathbf{R}_+^N$, then the class $\mathcal{A} = \left\{[0,t] : t \in \mathbf{R}_+^N\right\}$ satisfies all the assumptions. More generally, we can allow $\mathcal{A}$ to consist of all the *lower layers* of $\mathbf{R}_+^N$: a set $A$ is a lower layer if $[0,t] \subseteq A, \ \forall t \in A$.

Now, let $(\Omega, \mathcal{F}, P)$ be any complete probability space. A filtration (indexed by $\mathcal{A}$) is a class $\{\mathcal{F}_A : A \in \mathcal{A}\}$ of complete sub-$\sigma$-fields of $\mathcal{F}$ which satisfies the following conditions:

- $\forall A, B \in \mathcal{A}$, $\mathcal{F}_A \subseteq \mathcal{F}_B$ , if $A \subseteq B$.
- Monotone outer-continuity: $\mathcal{F}_{\bigcap A_i} = \bigcap \mathcal{F}_{A_i}$ for any decreasing sequence $(A_i)$ in $\mathcal{A}$.

**Definition 2.2** *A ($\mathcal{A}$ -indexed) stochastic process $X = \{X_A : A \in \mathcal{A}\}$ is a collection of random variables indexed by $\mathcal{A}$ with $X_\emptyset = 0$, and is said to be adapted if $X_A$ is $\mathcal{F}_A$-measurable, for every $A \in \mathcal{A}$. $X$ is said to be integrable (square integrable) if $E[|X_A|] < \infty$ ($E[(X_A)^2] < \infty$) for every $A \in \mathcal{A}$.*

**Remark 2.3** *Any multiparameter process $\tilde{X}$ can be considered as a set-indexed process $X$, setting $\mathcal{T} = \boldsymbol{R}_+^N$, $\mathcal{A} = \left\{[0,t];\ t \in \boldsymbol{R}_+^N\right\}$ and $\tilde{X}_t = X_{[0,t]}$.*

**Definition 2.4** *A ($\mathcal{A}$-indexed) stochastic process $X$ is* additive *if it has an (almost sure) additive extension to $\mathcal{C}(u)$: i.e., $X_\emptyset = 0$ and if $C, C_1, C_2 \in \mathcal{C}(u)$ with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$, then almost surely*

$$X_{C_1} + X_{C_2} = X_C.$$

In addition to assuming that $X_\emptyset = 0$, to avoid technicalities we will generally assume as well that $X_{\emptyset'} = 0$, where $\emptyset' := \bigcap_{A \in \mathcal{A}} A$.

**Definition 2.5** *An additive process $X$ is increasing if $X_C \geq 0\ \forall C \in \mathcal{C}$ and if for any decreasing sequence $(A_n)$ in $\mathcal{A}$, $X(\bigcap_n A_n) = \lim_n X(A_n)$.*

It is observed in Corollary 1.4.11 of [7] that an increasing process in fact defines a measure on $\mathcal{B}$ for each $\omega \in \Omega$.

**Definition 2.6** *Let $\Lambda$ be a non-negative increasing function defined on $\mathcal{A}$ with $\Lambda_\emptyset = 0$. We say that an $\mathcal{A}$-indexed additive process $X$ is a Brownian motion with variance measure $\Lambda$ if $X_\emptyset = 0$, and if for disjoint sets $C_1, \ldots, C_n \in \mathcal{C}$, $X_{C_1}, \ldots, X_{C_n}$ are independent mean-zero Gaussian random variables with variances $\Lambda_{C_1}, \ldots \Lambda_{C_n}$, respectively.*

## 2.2 Increments of a set-indexed process

The notion of increments for a set-indexed process $X = \{X_U;\ U \in \mathcal{A}\}$ is not as simple as in the case of real indices, where it is only the difference between values of the process.

In the case of multiparameter processes, we use to define the increment between $s, t \in \mathbf{R}_+^N$ by

$$\Delta X_{s,t} = \sum_{r \in \{0,1\}^N} (-1)^{N - \sum_i r_i} X_{[s_i + r_i(t_i - s_i)]_i} \tag{1}$$

which is different from the simple difference $X_t - X_s$ (see [4]).

In the case of set-indexed processes, the increments are defined from the collection of subsets $\mathcal{C}$.

For all $C = U \setminus \bigcup_{1 \leq i \leq n} U_i$, we define the increment of the process $X$ on $C$ by

$$\Delta X_C = X_U - \sum_{i=1}^{n} X_{U \cap U_i} + \sum_{i<j} X_{U \cap (U_i \cap U_j)} - \cdots + (-1)^n X_{U \cap (\bigcap_{1 \leq i \leq n} U_i)} \tag{2}$$

According to remark 2.3, this expression, applied to the multiparameter case, gives the definition (1) of the increments.

In the following, it would be important to consider the particular increments $\mathcal{C}_0 = \{C = U \setminus V;\ U, V \in \mathcal{A}\}$. Moreover the definition of the increment process $\Delta X$ can be extended to $\mathcal{C}(u)$, the finite unions of elements of $\mathcal{C}$. Particularly, for all $U, V \in \mathcal{A}$, $\Delta X_{U \triangle V}$ is well-defined.

**Remark 2.7** *The process* $\Delta X$ *could be seen as an extension of the process* $X$ *for the set of indices* $\mathcal{C}$. *For all* $A \in \mathcal{A} \subset \mathcal{C}$, *we have* $\Delta X_A = X_A$ *(because* $A = A \setminus \emptyset$*).*
*In the case of an additive process, the definition of the increment* $\Delta X_C$ *coincides with the additive extension of* $X_C$ *to* $C \in \mathcal{C}$. *However, if* $X$ *is not additive, which is the case of a direct definition of the process for the set of indices* $\mathcal{C}$, *in general* $\Delta X_C \neq X_C$ *for* $C \in \mathcal{C}$. *For this reason, we use a different notation for the increments of* $X$.

**Remark 2.8** *If* $X = \{X_U;\ U \in \mathcal{A}\}$ *is Gaussian, then* $\Delta X = \{\Delta X_C;\ C \in \mathcal{C}\}$ *is clearly Gaussian.*

## 2.3 Definition of sifBm

Recall that the fractional Brownian motion is defined to be a mean-zero Gaussian process such that

$$\forall s,t \in \mathbf{R}_+; \quad E\left[X_t - X_s\right]^2 = |t-s|^{2H}$$

The natural set-indexed extension of this process is to substitute the term $|t-s|^{2H}$ with $d(U,V)^{2H}$, where $d$ is some distance between two subsets of $\mathcal{T}$. In this paper, we consider the choice of $d(U,V) = m(U \bigtriangleup V)$, where $\bigtriangleup$ is the symmetric difference between two sets and $m$ is a measure on $\mathcal{T}$.

**Lemma 2.9** *Let* $m$ *be a finite measure on* $\mathcal{T}$. *For all* $\alpha \in (0,2]$, *the function*

$$(U,V) \mapsto m(U)^\alpha + m(V)^\alpha - m(U \bigtriangleup V)^\alpha$$

*is positive definite.*

**Proof** For all measurable subset $U$ of $\mathcal{T}$ such that $m(U) < +\infty$, we define the elementary function $f = \mathbb{1}_U$. Finite linear combinations of elementary functions are called simple functions. It is well-known that simple functions are dense in $L^2(m)$.
Moreover, for all $U,V \subseteq \mathcal{T}$, we have

$$\begin{aligned}
\mathbb{1}_{U \triangle V} &= \mathbb{1}_{(U\setminus V)\cup(V\setminus U)} \\
&= \mathbb{1}_U(1-\mathbb{1}_V) + \mathbb{1}_V(1-\mathbb{1}_U) \\
&= \mathbb{1}_U + \mathbb{1}_V - 2\ \mathbb{1}_U.\mathbb{1}_V \\
&= (\mathbb{1}_U - \mathbb{1}_V)^2 \\
&= |\mathbb{1}_U - \mathbb{1}_V|
\end{aligned}$$

Then we only have to show that the function

$$\begin{array}{rcl}
L^2(m) \times L^2(m) & \to & \mathbf{R} \\
(f,g) & \mapsto & m(f^2)^\alpha + m(g^2)^\alpha - m(|f-g|^2)^\alpha
\end{array}$$

is positive definite.
Let $f_1, f_2, \ldots, f_n \in L^2(m)$ and $u_1, u_2, \ldots, u_n \in \mathbf{R}$. We have to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \left\{ m(f_i^2)^\alpha + m(f_j^2)^\alpha - m(|f_i - f_j|^2)^\alpha \right\} u_i u_j \geq 0 \tag{3}$$

Setting $u_0 = -\sum_{i=1}^n u_i$ and $f_0 = \mathbb{1}_\emptyset$, we can write

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{m(f_i^2)^\alpha + m(f_j^2)^\alpha - m(|f_i - f_j|^2)^\alpha\right\} u_i u_j = -\sum_{i=0}^{n}\sum_{j=0}^{n} m(|f_i - f_j|^2)^\alpha u_i u_j \tag{4}$$

But, for all $\lambda > 0$, we have

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\lambda m(|f_i-f_j|^2)^\alpha} u_i u_j &= \sum_{i=0}^{n}\sum_{j=0}^{n}\left(e^{-\lambda m(|f_i-f_j|^2)^\alpha} - 1\right) u_i u_j \\ &= -\lambda \sum_{i=0}^{n}\sum_{j=0}^{n} m(|f_i - f_j|^2)^\alpha u_i u_j + o(\lambda) \end{aligned} \tag{5}$$

Then, (3) is equivalent to

$$\sum_{i=0}^{n}\sum_{j=0}^{n} e^{-\lambda m(|f_i-f_j|^2)^\alpha} u_i u_j \geq 0 \tag{6}$$

in the neighborhood of $\lambda = 0$.

In $L^2(m)$, let us define the bilinear form $< f, g >= m(f.g)$. If we identify the elements of $L^2(m)$ that are almost everywhere equal, then $L^2(m)$ is a complete separable metric space with this scalar product.
Let us show that there exists a random variable $X$ taking its values in $L^2(m)$, such that

$$\forall f \in L^2(m); \quad E\left[e^{i<f,X>}\right] = e^{-\lambda\|f\|^{2\alpha}} \tag{7}$$

Consider a $\alpha$-stable real random variable

$$Y \overset{(d)}{=} S_\alpha\left(\left(\cos\frac{\pi\alpha}{2}\right)^{\frac{1}{\alpha}}, 1, 0\right)$$

where $0 < \alpha \leq 2$ and $\alpha \neq 1$ (see [13], prop. 1.2.12, p 15).
As the function $f \mapsto e^{-\frac{1}{2}\|f\|^2}$ is positive definite (see remark 2.10), by a theorem of Bochner-Minlos, there exists a random variable $G$, such that

$$\forall f \in L^2(m); \quad E\left[e^{i<f,G>}\right] = e^{-\frac{1}{2}\|f\|^2}.$$

Moreover, we can suppose that $G$ is independant from $Y$.
For all $f \in L^2(m)$, we compute

$$\begin{aligned} E\left[e^{i<f,Y^{1/2}G>}\right] &= E\left\{E\left[e^{i<f,Y^{1/2}G>} \mid Y\right]\right\} \\ &= E\left\{E\left[e^{iY^{1/2}<f,G>} \mid Y\right]\right\} \\ &= E\left[e^{-\frac{Y}{2}\|f\|^2}\right] \end{aligned}$$

As $E\left[e^{-\gamma Y}\right] = e^{-\gamma^\alpha}$ for all $\gamma > 0$, we get

$$E\left[e^{i<f,Y^{1/2}G>}\right] = e^{-2^{-\alpha}\|f\|^{2\alpha}}$$

Then $X = \begin{cases} \sqrt{2\lambda^{1/\alpha}}\, Y^{1/2} G & \text{if } \alpha \neq 1 \\ \sqrt{2\lambda}\, G & \text{if } \alpha = 1 \end{cases}$ satisfies (7), and $f \mapsto e^{-\lambda\|f\|^{2\alpha}}$ is non-negative definite. That proves (6) and the result follows. □

**Remark 2.10** *For all family* $(f_1, \ldots, f_n)$ *of* $L^2(m)$ *provided with the scalar product previously defined, there exist* $p \leq n$ *and a family* $(x_1, \ldots, x_n)$ *of* $\boldsymbol{R}^p$ *such that*

$$\forall i, j; \quad \|x_i - x_j\|_{\boldsymbol{R}^p} = \|f_i - f_j\|_{L^2(m)} \tag{8}$$

To show this result, let us consider an orthonormal basis $(e_1, \ldots, e_p)$ of $Vect(f_i)_{1 \leq i \leq n}$, and the canonical basis $(\epsilon_1, \ldots, \epsilon_p)$ of $\mathbf{R}^p$. For all $i$, there exists a family $(\lambda_1^i, \ldots, \lambda_p^i)$ s.t. $f_i = \sum_k \lambda_k^i . e_k$. Then the vectors $(x_1, \ldots, x_n)$ defined by $x_i = \sum_k \lambda_k^i . \epsilon_k$ satisfy (8). □
This remark allows to show directly (3) using lemma 2.10.8 in [13]

Since the existence of a mean-zero Gaussian process is equivalent to the positive definite property of its covariance function, we can define

**Definition 2.11** *A mean-zero Gaussian process* $B^H = \{B_U^H;\ U \in \mathcal{A}\}$ *satisfying*

$$E\left[B_U^H B_V^H\right] = \frac{1}{2}\left[m(U)^{2H} + m(V)^{2H} - m(U \bigtriangleup V)^{2H}\right] \tag{9}$$

*where* $H \in (0, 1)$, *is called a set-indexed fractional Brownian motion (sifBm).* $H$ *is the index of self-similarity of the process.*

**Remark 2.12** *If* $H = \frac{1}{2}$, *the process* $B^{\frac{1}{2}}$ *is the well known set-indexed Brownian motion. Indeed, let us compute the covariance function of this process*

$$E\left[B_U^{\frac{1}{2}} B_V^{\frac{1}{2}}\right] = \frac{1}{2}\left[m(U) + m(V) - m(U \bigtriangleup V)\right]$$

*As*

$$\begin{aligned} m(U \bigtriangleup V) &= m(U \setminus V) + m(V \setminus U) \\ &= m(U \setminus U \cap V) + m(V \setminus U \cap V) \\ &= m(U) + m(V) - 2\, m(U \cap V) \end{aligned}$$

*we have*

$$E\left[B_U^{\frac{1}{2}} B_V^{\frac{1}{2}}\right] = m(U \cap V)$$

*which is the covariance function of the set-indexed Brownian motion.*

**Remark 2.13** *In the case of* $\mathcal{T} = \boldsymbol{R}_+$ *and* $\mathcal{A} = \{[0, t];\ t \in \boldsymbol{R}_+\}$, *the process* $B^H$ *is the classical fBm. Indeed, the covariance function is*

$$E\left[B_{[0,s]}^H B_{[0,t]}^H\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |t - s|^{2H}\right]$$

*which is the covariance function of the fractional Brownian motion.*

**Remark 2.14** *In the case of* $\mathcal{T} = \boldsymbol{R}_+^N$ *and* $\mathcal{A}$ *is the set of rectangles of the form* $[0, t]$, *the process* $B^H$ *can be seen as a multiparameter process. Then it is interesting to compare it with the other known multiparameter fractional Brownian motions (see [4]).*

- *the Lévy fractional Brownian motion is a mean-zero Gaussian process* $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ *such that*

$$\forall s,t \in \boldsymbol{R}_+^N; \quad E\left[X_s X_t\right] = \frac{1}{2}\left[\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right]$$

*where* $H \in (0,1)$*.*

- *the fractional Brownian sheet is a mean-zero Gaussian process* $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ *such that*

$$\forall s,t \in \boldsymbol{R}_+^N; \quad E\left[X_s X_t\right] = \frac{1}{2}\prod_{i=1}^{N}\left[s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}\right]$$

*where* $H = (H_1, \ldots, H_N) \in (0,1)^N$*.*

*As for all* $s,t \in \boldsymbol{R}_+^N$*,*

$$E\left[B^H_{[0,s]} B^H_{[0,t]}\right] = \frac{1}{2}\left[\prod_{i=1}^{N} s_i^{2H} + \prod_{i=1}^{N} t_i^{2H} - \left(\prod_i s_i + \prod_i t_i - 2\prod_i s_i \wedge t_i\right)^{2H}\right]$$

*we can see that* $B^H$ *is different from these two processes.*

This fact will be also shown in the next sections in the study of properties of sifbm, and its restriction on flows. It is therefore natural to wonder if the Levy fBm and the fBs can have set-indexed extension. The answer seems to be negative.
Actually the definition of the sheet is strongly associated with the Euclidean structure of $\mathbf{R}^N$. Therefore it is incompatible with a set-indexed viewpoint.
Moreover the Levy fBm can be seen as a simple one parameter process where the increment between two points only depends from distance between them.

# 3 Fractal properties

The fractional Brownian motion has two important properties which make it the most natural fractal process:

- its increments are stationary

$$\forall h \in \mathbf{R}_+; \quad (X_{t+h} - X_h)_{t\in\mathbf{R}_+} \stackrel{(d)}{=} (X_t - X_0)_{t\in\mathbf{R}_+}$$

- it is self-similar

$$\forall a \in \mathbf{R}_+; \quad (X_{at})_{t\in\mathbf{R}_+} \stackrel{(d)}{=} \left(a^H X_t\right)_{t\in\mathbf{R}_+}$$

Moreover, the fBm is the only Gaussian process which has these two properties.

In this section, we show that in some sense these properties still hold for the set-indexed fractional Brownian motion. Moreover they characterize the covariance function of the process between two sets $U$ and $V$ such that $U \subseteq V$.

## 3.1 Stationarity of the increments

Stationarity of increments of a set-indexed process can be defined in various ways. The set-indexed Brownian motion satisfies all of them, but the different extensions of *fractional* Brownian motion do not.

In the case of $\mathcal{T} = \mathbf{R}_+^N$ and $\mathcal{A}$ is the collection of rectangles, the classical definition of stationarity of increments can be studied.
A process $X = \left\{X_{[0,t]};\ t \in \mathbf{R}_+^N\right\}$ is said to have *stationary increments against translations* if for all $\tau \in \mathbf{R}_+^N$, the two processes $\left\{\Delta X_{[\tau,t+\tau]};\ t \in \mathbf{R}_+^N\right\}$ and $\left\{\Delta X_{[0,t]};\ t \in \mathbf{R}_+^N\right\}$ have the same law.
Both Lévy fractional Brownian motion and fractional Brownian sheet satisfy this property of stationarity (see [4]).

**Remark 3.1** *This definition is weaker than stationarity of increments against isometries of $\boldsymbol{R}_+^N$, i.e. for all $g \in \mathcal{G}(\boldsymbol{R}^N)$,*

$$\left\{\Delta X_{g([0,t])};\ t \in \boldsymbol{R}_+^N\right\} \overset{(d)}{=} \left\{\Delta X_{[0,t]};\ t \in \boldsymbol{R}_+^N\right\}$$

*where $\mathcal{G}(\boldsymbol{R}^N)$ is the group of isometries of $\boldsymbol{R}^N$ preserving $\mathcal{C}$.*
*On the contrary to stationarity against translations, the context of set-indexed processes imposes additional assumptions to the strict context of multiparameter processes. Actually as the image of $C \in \mathcal{C}$ by any isometry of $\boldsymbol{R}^N$ does not necessarily belong to $\mathcal{C}$, a stability assumption is needed for the definition to make sense.*
*However in the strict context of multiparameter processes, this assumption does not need to be considered. The Lévy fractional Brownian motion satisfy this property of increment stationarity in the strong sense (see [13]).*

However in general, there is no reason that the SifBm possesses the stationarity increments property against translations. In some particular cases, we can show directly using the next lemma that this property is not satisfied.

**Lemma 3.2** *Let $B^H = \left\{B_U^H;\ U \in \mathcal{A}\right\}$ be a SifBm of index $H \in (0,1)$. For all $C = U \setminus \left(\bigcup_{1\le i\le n} U_i\right) \in \mathcal{C}$, where $\forall i \in \{1,\ldots,n\};U_i \subset U$, we have*

$$\begin{aligned}
E\left[\Delta B_C^H\right]^2 &= -\sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} m\left(U \bigtriangleup \left[\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right]\right)^{2H} \\
&+ \frac{1}{2}\sum_{k,l}(-1)^{k+l} \sum_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_l}} m\left(\left[\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right] \bigtriangleup \left[\bigcap_{p\in\{j_1,\ldots,j_l\}} U_p\right]\right)^{2H} \qquad (10)
\end{aligned}$$

**Proof** By definition,

$$\Delta B_C^H = B_U^H + \sum_{k=1}^{n}(-1)^k \sum_{i_1<\cdots<i_k} B^H_{\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p}$$

Then,

$$E\left[\Delta B_C^H\right]^2 = E\left[B_U^H\right]^2 + 2\sum_k (-1)^k \sum_{i_1<\cdots<i_k} E\left[B_U^H . B^H_{\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p}\right]$$
$$+\sum_{k,l}(-1)^{k+l} \sum_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_l}} E\left[B^H_{\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p} . B^H_{\bigcap_{p\in\{j_1,\ldots,j_l\}} U_p}\right]$$

and, using the covariance function of $X$

$$E\left[\Delta B_C^H\right]^2 = m(U)^{2H}$$
$$+\sum_k (-1)^k \sum_{i_1<\cdots<i_k} \left\{ m(U)^{2H} + m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H} - m\left(U \triangle \left[\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right]\right)^{2H} \right\}$$
$$+\frac{1}{2}\sum_{k,l}(-1)^{k+l} \sum_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_l}} \left\{ m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H} + m\left(\bigcap_{p\in\{j_1,\ldots,j_l\}} U_p\right)^{2H} \right.$$
$$\left. -m\left(\left[\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right] \triangle \left[\bigcap_{p\in\{j_1,\ldots,j_l\}} U_p\right]\right)^{2H} \right\} \quad (11)$$

Let us consider the two following terms in expression (11):

- term in $m(U)^{2H}$

$$m(U)^{2H} + \sum_k (-1)^k \sum_{i_1<\cdots<i_k} m(U)^{2H} = \sum_{k=0}^{n} C_n^k \, m(U)^{2H} = 0$$

- term in $m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H}$

$$\sum_k (-1)^k \sum_{i_1<\cdots<i_k} m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H} + \underbrace{\sum_{k,l}(-1)^{k+l} \sum_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_l}} m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H}}_{\sum_k (-1)^k \sum_{i_1<\cdots<i_k} m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H} \sum_{l=1}^{n}(-1)^l C_n^l}$$

therefore this term is equal to

$$\sum_k (-1)^k \sum_{i_1<\cdots<i_k} m\left(\bigcap_{p\in\{i_1,\ldots,i_k\}} U_p\right)^{2H} \sum_{l=0}^{n}(-1)^l C_n^l = 0$$

The two other terms of expression (11) give the result. □

The main idea to define a set-indexed extension of the fractional Brownian motion, was to extend

$$\forall s,t \in \mathbf{R}_+; \quad E\left[X_t - X_s\right]^2 = |t-s|^{2H}$$

in

$$\forall U,V \in \mathcal{A}; \quad E\left[X_U - X_V\right]^2 = m(U \triangle V)^{2H}$$

However, it should be more interesting to get

$$\forall C \in \mathcal{C}; \quad E\left[\Delta X_C\right]^2 = m(C)^{2H} \tag{12}$$

According to lemma 3.2, the set-indexed fractional Brownian motion $B^H$ satisfies

$$\forall U,V \in \mathcal{A}; \quad E\left[\Delta B^H_{U\setminus V}\right]^2 = m(U \setminus V)^{2H}$$

but the property (12) does not hold.
Moreover, we will see that there is no set-indexed process satisfying (12) for $H \neq \frac{1}{2}$ (theorem 4.3).

**Proposition 3.3** *Let $X = \left\{X_t;\ t \in \boldsymbol{R}^N_+\right\}$ be a fractional Brownian sheet of constant parameter $H$ in every axis.*
*For all $a,b \in \boldsymbol{R}^N_+$ such that $a \prec b$ (i.e. $\forall i = 1,\ldots,N;\ a_i < b_i$), we have*

$$E\left[\Delta X_{[a,b]}\right]^2 = m([a,b])^{2H}$$

*and consequently, For all $a,b,a',b' \in \boldsymbol{R}^N_+$, such that $m([a,b]) = m([a',b'])$, we have*

$$\Delta X_{[a,b]} \overset{(d)}{=} \Delta X_{[a',b']}$$

**Proof** For all $a,b \in \mathbf{R}^N_+$ with $a \prec b$, as $X$ has stationary increments against translations, we have

$$\begin{aligned} E\left[\Delta X_{[a,b]}\right]^2 &= E\left[\Delta X_{[0,b-a]}\right]^2 \\ &= E\left[X_{b-a}\right]^2 \\ &= \prod_{i=1}^{N} |b_i - a_i|^{2H} \\ &= m([a,b])^{2H} \end{aligned}$$

As $\Delta X$ is a mean-zero Gaussian process, the result follows. □

**Remark 3.4**
- *If a fractional Brownian sheet of index $H = (H_1,\ldots,H_N)$ satisfies the property of proposition 3.3, we have $H_1 = \cdots = H_N$.*
- *The Levy fractional Brownian motion does not satisfy this property.*

**Definition 3.5** *A set-indexed process $X$ is said to be $\mathcal{C}_0$-increment stationary if for all $C,C' \in \mathcal{C}_0$ such that $m(C) = m(C')$, we have $\Delta X_C \overset{(d)}{=} \Delta X_{C'}$.*

In the case of one-parameter fractional Brownian motion, the increment stationarity property gives an equality between laws of some increment processes. However, in the set-indexed case, definition 3.5 does not tell anything about correlation between increments.
In section 4, we see that a stronger property is not worth to be considered.

**Proposition 3.6** *The set-indexed fractional Brownian motion $B^H$ is $\mathcal{C}_0$-increment stationary.*

**Proof** For all $C \in \mathcal{C}_0$, there exist $U, V \in \mathcal{A}$ where $V \subset U$, such that $C = U \setminus V$. Then $\Delta B^H_C = B^H_U - B^H_{U \cap V} = B^H_U - B^H_V$. We compute

$$\begin{aligned} E\left[\Delta B^H_C\right]^2 &= E\left[B^H_U - B^H_V\right]^2 \\ &= m(U \bigtriangleup V)^{2H} \\ &= m(C)^{2H} \end{aligned}$$

Thus, as $\Delta B^H$ is a Gaussian process, for all $C, C' \in \mathcal{C}_0$ such that $m(C) = m(C')$, we have $\Delta B^H_C \stackrel{(d)}{=} \Delta B^H_{C'}$. $\square$

**Remark 3.7** *In the proof of proposition 3.6, we saw that*

$$\forall C \in \mathcal{C}_0;\ E\left[\Delta B^H_C\right]^2 = m(C)^{2H}$$

*However, in general*

$$E\left[\Delta B^H_C . \Delta B^H_{C'}\right] \neq \frac{1}{2}\left[m(C)^{2H} + m(C')^{2H} - m(C \bigtriangleup C')^{2H}\right]$$

*In fact, it can be shown that for $C = U \setminus V$ where $V \subset U$, and $C' = U' \setminus V'$ where $V' \subset U'$, and $U, V, U', V' \in \mathcal{A}$*

$$E\left[\Delta B^H_C . \Delta B^H_{C'}\right] = \frac{1}{2}\left[m(U \bigtriangleup V')^{2H} + m(V \bigtriangleup U')^{2H} - m(U \bigtriangleup U')^{2H} - m(V \bigtriangleup V')^{2H}\right]$$

## 3.2 Self-similarity

To study a set-indexed version of the notion of self-similarity for a set-indexed process, we need some assumptions about the set $\mathcal{A}$.

We suppose that $\mathcal{A}$ is provided with the operation of a group $G$ that can be extended satifying

$$\forall U, V \in \mathcal{A}, \forall g \in G; \quad \begin{aligned} g.(U \cup V) &= g.U \cup g.V \\ g.(U \setminus V) &= g.U \setminus g.V \end{aligned} \tag{13}$$

and assume there exists a function $\mu : G \to \mathbf{R}_+$

$$\forall U \in \mathcal{A}, \forall g \in G; \quad m(g.U) = \mu(g).m(U) \tag{14}$$

**Remark 3.8** *We can see easily that $\mu$ is an group-homomorphism.*

**Example 3.9** *In the case of* $\mathcal{T} = \boldsymbol{R}^N$ *and* $\mathcal{A} = \left\{[0,t];\ t \in \boldsymbol{R}^N\right\}$, *we can consider the multiplication by the elements of* $\boldsymbol{R}_+$

$$\forall g \in \boldsymbol{R}_+, \forall t \in \boldsymbol{R}^N; \quad g.[0,t] = [0, g.t]$$

*Moreover,*

$$\forall g \in \boldsymbol{R}_+, \forall t \in \boldsymbol{R}^N; \quad m(g.[0,t]) = g^N m([0,t])$$

The following result will be useful in the next section.

**Lemma 3.10** *Under the assumptions about the group* $G$, *the cardinal of* $G$ *is not finite.*

**Proof** If the function $\mu$ is not constant, there exists $g \in G$ such that $\mu(g) > 1$ (take $\tilde{g}$ s.t. $\mu(g) \neq 1$ and then $g = \tilde{g}$ or $g = \tilde{g}^{-1}$).
For all integer $n$, we have $\mu(g^n) = [\mu(g)]^n$. If $G$ is finite, the set $\{g^n;\ n \in \mathbf{N}\}$ is finite, which is in conflict with $\lim_{n\to\infty} \mu(g^n) = \infty$. □

**Definition 3.11** *A set-indexed process* $X = \{X_U;\ U \in \mathcal{A}\}$ *is said to be self-similar of index* $H$, *if for all* $g \in G$,

$$\{X_{g.U};\ U \in \mathcal{A}\} \stackrel{(d)}{=} \{\mu(g)^H.X_U;\ U \in \mathcal{A}\} \tag{15}$$

**Proposition 3.12** *The set-indexed fractional Brownian motion* $B^H$ *is self-similar of index* $H$, *for the operation of the group* $G$.

**Proof** Let $g$ be an element of the group $G$. For all $U, V \in \mathcal{A}$, we have

$$E\left[B^H_{g.U} B^H_{g.V}\right] = \frac{1}{2}\left[m(g.U)^{2H} + m(g.V)^{2H} - m(g.U \bigtriangleup g.V)^{2H}\right]$$

As $g.(U \bigtriangleup V) = g.U \bigtriangleup g.V$, we get

$$\begin{aligned} E\left[B^H_{g.U} B^H_{g.V}\right] &= \frac{\mu(g)^{2H}}{2}\left[m(U)^{2H} + m(V)^{2H} - m(U \bigtriangleup V)^{2H}\right] \\ &= \mu(g)^{2H} E\left[B^H_U B^H_V\right] \end{aligned}$$

Therefore, the two mean-zero Gaussian processes $\left\{B^H_{g.U};\ U \in \mathcal{A}\right\}$ and $\left\{\mu(g)^H.B^H_U;\ U \in \mathcal{A}\right\}$ have the same law. □

# 4 Pseudo-characterisation of SifBm

Recall that the fractional Brownian motion is the only mean-zero Gaussian process which is self-similar and has stationary increments. In the same way, the only multiparameter mean-zero Gaussian process which is self-similar and whose increments are stationary in the strong sense (under isometries of $\mathbf{R}^N$), is the Levy fractional Brownian motion ([13]).
In the case of the set-indexed processes, there is not such a characterisation. However, the two properties of self-similarity and stationarity of increments characterise the covariance function of the process between all $U$ and $V$ such that $U \subseteq V$.

**Proposition 4.1** *Let $X = \{X_U;\ U \in \mathcal{A}\}$ be a set-indexed process satisfying the two following properties*

1. *self-similarity of index $H$*
2. *$\mathcal{C}_0$-increment stationarity*

*Then, the covariance function between two subsets $U$ and $V$ such that $U \subseteq V$ is*

$$E[X_U X_V] = K\ \left[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}\right] \tag{16}$$

**Proof** Let $U_0$ be a non $m$-null fixed element of $\mathcal{A}$. For all $U, V \in \mathcal{A}$ such that $U \subset V$, there exists $g \in G$ such that $m(g.U_0) = m(V \setminus U)$, i.e. $\mu(g).m(U_0) = m(V \setminus U)$. Then using $\mathcal{C}_0$-increment stationarity property, we have

$$\begin{aligned} E[X_V - X_U]^2 &= E\left[\Delta X_{V\setminus U}\right]^2 \\ &= E\left[\Delta X_{g.U_0}\right]^2 \end{aligned}$$

As $g.U_0 \in \mathcal{A}$, we have $\Delta X_{g.U_0} = X_{g.U_0}$ and by self-similarity

$$\begin{aligned} E[X_V - X_U]^2 &= E\left[X_{g.U_0}\right]^2 \\ &= [\mu(g)]^{2H}\, E\left[X_{U_0}\right]^2 \\ &= [m(V \setminus U)]^{2H}\, \frac{E\left[X_{U_0}\right]^2}{m(U_0)^{2H}} \end{aligned} \tag{17}$$

The result follows from (17). □

**Remark 4.2** *The proposition 4.1 shows that our set-indexed extension of the fractional Brownian motion is very natural provided the two properties of self-similarity and stationarity of the increments.*

*However, if there exists a mean-zero Gaussian process with covariance function*

$$E[X_U X_V] = \frac{1}{2}\left[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H} - m(U \setminus V)^{2H}\right] \tag{18}$$

*it satisfies proposition 4.1 as well.*

To determine completely the covariance function of a self-similar, $\mathcal{C}_0$-increments, set-indexed process, we need assumptions about $E\left[\Delta X_{U\setminus V}.\Delta X_{V\setminus U}\right]$, where $U, V \in \mathcal{A}$.
For all $U, V \in \mathcal{A}$, we have $\Delta X_{U\setminus V} = X_U - X_{U\cap V}$ and $\Delta X_{V\setminus U} = X_V - X_{U\cap V}$.
Then,

$$\begin{aligned} E\left[\Delta X_{U\setminus V}.\Delta X_{V\setminus U}\right] &= E[X_U.X_V] - E[X_U.X_{U\cap V}] - E[X_V.X_{U\cap V}] + E[X_{U\cap V}]^2 \\ &= E[X_U.X_V] - \frac{1}{2}\left[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H} - m(U \setminus V)^{2H}\right] \end{aligned}$$

Particularly, assuming the independance of $\Delta X_{U\setminus V}$ and $\Delta X_{V\setminus U}$, is equivalent to (18), provided that such a process $X$ exists.

The property of $\mathcal{C}_0$-increment stationarity seems too weak to characterize completely the covariance of a self-similar process. It can be tempting to define a process which would have a stronger kind of increment stationarity. For instance, does it exist a self-similar process which satisfy $E[\Delta X_C]^2 = m(C)^{2H}$ for all $C \in \mathcal{C}$?

**Theorem 4.3** *The only Gaussian set-indexed process $X = \{X_U;\ U \in \mathcal{A}\}$ such that*

$$\forall C \in \mathcal{C}; \quad E[\Delta X_C]^2 = K.m(C)^{2H} \tag{19}$$

*where $K > 0$ and $H \in (0,1)$, is the set-indexed Brownian motion.*

**Proof** Let $X$ be a set-indexed Gaussian process satisfying (19).
First, we can see that $X$ is $\mathcal{C}_0$-increment stationary. Moreover, for all $U \in \mathcal{A}$ and $g \in G$, we have $E[X_{g.U}]^2 = K.\mu(g)^{2H}.m(U)^{2H} = \mu(g)^{2H}.E[X_U]^2$. Then, as $X$ is Gaussian, we conclude that $X$ is self-similar. Proposition 4.1 implies that for all $U, V \in \mathcal{A}$, such that $U \subset V$,

$$E[X_U.X_V] = K\left[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}\right] \tag{20}$$

For all $U_1$ and $U_2$ in $\mathcal{A}$, let us consider $U \in \mathcal{A}$ such that $U_1 \subset U$ and $U_2 \subset U$. The subset of $\mathcal{T}$, $C = U \setminus (U_1 \cup U_2)$ belongs to $\mathcal{C}$ and $\Delta X_C = X_U - X_{U_1} - X_{U_2} + X_{U_1 \cap U_2}$. Then we have

$$\begin{aligned} E[\Delta X_C]^2 =& E[X_U]^2 + E[X_{U_1}]^2 + E[X_{U_2}]^2 + E[X_{U_1\cap U_2}]^2 \\ & - 2E[X_U.X_{U_1}] - 2E[X_U.X_{U_2}] + 2E[X_U.X_{U_1\cap U_2}] \\ & + 2E[X_{U_1}.X_{U_2}] - 2E[X_{U_1}.X_{U_1\cap U_2}] - 2E[X_{U_2}.X_{U_1\cap U_2}] \end{aligned}$$

Hence

$$\begin{aligned} 2E[X_{U_1}.X_{U_2}] = & E[\Delta X_C]^2 - E[X_U]^2 - E[X_{U_1}]^2 - E[X_{U_2}]^2 - E[X_{U_1\cap U_2}]^2 \\ & + 2E[X_U.X_{U_1}] + 2E[X_U.X_{U_2}] - 2E[X_U.X_{U_1\cap U_2}] \\ & + 2E[X_{U_1}.X_{U_1\cap U_2}] + 2E[X_{U_2}.X_{U_1\cap U_2}] \end{aligned}$$

Using (20), we get

$$\begin{aligned} 2E[X_{U_1}.X_{U_2}] =& K\left\{m(U \setminus (U_1\cup U_2))^{2H} - m(U\setminus U_1)^{2H} - m(U\setminus U_2)^{2H} + m(U\setminus(U_1\cap U_2))^{2H}\right\} \\ & + K\left[m(U_1)^{2H} + m(U_2)^{2H} - m(U_1\setminus U_2)^{2H} - m(U_2\setminus U_1)^{2H}\right] \end{aligned} \tag{21}$$

Taking $V \in \mathcal{A}$ such that $U \subsetneq V$, we get an expression of $2.E[X_{U_1}.X_{U_2}]$ different from (21) if $H \neq \frac{1}{2}$. □

**Corollary 4.4** *There exists no set-indexed process which is $H$-self-similar (for $H \neq \frac{1}{2}$) and whose increments satisfy one of the following*

1. *$\mathcal{C}$-increment stationarity*

$$\forall C, C' \in \mathcal{C}; \quad m(C) = m(C') \Rightarrow \Delta X_C \stackrel{(d)}{=} \Delta X_{C'}$$

2. *for all function $f : \mathcal{C} \to \mathcal{C}$, such that $\forall C \in \mathcal{C}$; $m(f(C)) = m(C)$*

$$\left\{\Delta X_{f(C)};\ C \in \mathcal{C}\right\} \stackrel{(d)}{=} \left\{\Delta X_C;\ C \in \mathcal{C}\right\}$$

**Proof** First, we can see easily that the second property implies $\mathcal{C}$-increment stationarity. Then we only need to consider the first property.
Let $U_0$ be a fixed element of $\mathcal{A}$. For all $C \in \mathcal{C}$, there exists $g \in G$ such that $m(g.U_0) = m(C)$, i.e. $\mu(g).m(U_0) = m(C)$. Therefore, by $\mathcal{C}$-increment stationarity,

$$\begin{aligned} E\left[\Delta X_C\right]^2 &= E\left[X_{g.U_0}\right]^2 \\ &= \mu(g)^{2H} E\left[X_{U_0}\right]^2 \\ &= m(C)^{2H} \frac{E\left[X_{U_0}\right]^2}{m(U_0)^{2H}} \end{aligned}$$

By theorem 4.3, the result follows. □

# 5 Continuity of the SifBm

The results about the existence of a continuous version of set-indexed processes are not as simple as processes indexed by $\mathbf{R}_+$. Even in the simple case of the set-indexed Brownian motion, if the collection $\mathcal{A}$ is too rich, there does not exist any version which is continuous on the whole $\mathcal{A}$ (see [1]).

To study the continuity of a set-indexed process $X$, we have to consider the behavior of $|X_U - X_V|$ when $U$ and $V$ are close. In order to do this, we provide $\mathcal{A}$ with some distance. In the classical case of Gaussian processes, we used to consider the canonical distance $d^2(U,V) = E\left[X_U - X_V\right]^2$ for $U, V \in \mathcal{A}$ (see [1],[3]). Let us mention two other distances that are also classical:

- The measure $m$ on $\mathcal{T}$ induces the pseudo-metric $d_m$ on $\mathcal{A}$

$$\forall U, V \in \mathcal{A}; \quad d_m(U,V) = m(U \bigtriangleup V)$$

- we recall the definition of the Hausdorff metric $d_{Haus}$ on $\mathcal{K} \setminus \emptyset$, the nonempty compact subsets of $\mathcal{T}$

$$\forall U, V \in \mathcal{K} \setminus \emptyset; \quad d_{Haus}(U,V) = \inf\{\epsilon > 0 : U \subseteq V^\epsilon \text{ and } V \subseteq U^\epsilon\}$$

The notion of continuity depends of the distance considered. However, if $(\mathcal{A}, d_m)$ (resp.$(\mathcal{A}, d_{Haus})$) is compact, then $d$-continuity and $d_m$-continuity (resp. $d_{Haus}$-continuity) are equivalent (see [1]).

For any function $x : \mathcal{A} \to \mathbf{R}$, define

$$\|x\|_{\mathcal{A}} = \sup_{A \in \mathcal{A}} |x(A)|$$

and let

$$B(\mathcal{A}) = \{x : \mathcal{A} \to \mathbf{R}; \|x\|_{\mathcal{A}} < \infty\}.$$

Let $C(\mathcal{A})$ denote the class of functions in $B(\mathcal{A})$ which are $d_{Haus}$-continuous on $\mathcal{A}$.

Then, studying $d$-continuity, we consider the balls $\mathcal{B}(U,\epsilon) = \{V \in \mathcal{A};\ d(U,V) < \epsilon\}$, and the metric entropy $D(\bullet; \mathcal{A}, d)$ of $(\mathcal{A}, d)$, which gives the smallest number $D(\epsilon; \mathcal{A}, d)$ of balls of radius $\epsilon > 0$ required to cover $\mathcal{A}$.

If $(\mathcal{A}, d)$ is totally bounded, i.e. $D(\epsilon; \mathcal{A}, d)$ is finite for all $\epsilon > 0$, then under the assumption

$$\int_0^1 \sqrt{\ln D(\epsilon; \mathcal{A}, d)}.d\epsilon < \infty$$

Dudley's theorem states that the process $X$ has a continuous modification (see [1], [3], [9]).

**Theorem 5.1** *Let $B^H = \{B_U^H;\ U \in \mathcal{A}\}$ be a set-indexed fractional Brownian motion.*
*The two following statements are equivalent*

1. *$B^H$ is almost surely continuous on $\mathcal{A}$.*
2. *the set-indexed Brownian motion $W$ is almost surely continuous on $\mathcal{A}$.*

**Proof** As

$$\forall U, V \in \mathcal{A}; \quad E\left[B_U^H - B_V^H\right]^2 = \left(E\left[W_U - W_V\right]^2\right)^{2H} \tag{22}$$

the two canonical pseudo-metrics associated to the set-indexed fractional Brownian motion $B^H$ and the set-indexed Brownian motion $W$, are equivalent.
Then, Dudley's theorem gives the result. □

A simple consequence of this result is that the sifBm has a continuous modification on rectangles of $\mathbf{R}_+^N$.

# 6 SifBm on increasing paths

The notion of flows is the key to reducing the proofs of many of theorems on characterization and weak convergence to a one-dimensional problem (see [7]).

## 6.1 Generality on flows

In general, $\mathcal{A}(u)$ is not closed under countable intersections, so we will occasionally require a larger class $\tilde{\mathcal{A}}(u)$ , which is the class of countable intersections of sets in $\mathcal{A}(u)$: i.e. $U \in \tilde{\mathcal{A}}(u)$ if there exists a sequence $(U_n)_{n\in\mathbf{N}}$ in $\mathcal{A}(u)$ such that $\bigcap_n U_n = U$.

**Definition 6.1** *Let $S = [a, b] \subseteq \boldsymbol{R}$. An increasing function $f : S \to \tilde{\mathcal{A}}(u)$ is called a flow.*

- *A flow $f$ is right-continuous if*

$$f(s) = \bigcap_{v>s} f(v), \forall s \in [a, b),$$

*and $f(b) = \overline{\bigcup_{u<b} f(u)}$.*

- *A flow $f$ is continuous if it is right-continuous and*

$$f(s) = \overline{\bigcup_{u<s} f(u)}\ \forall s \in (a, b).$$

- *A flow $f$ is simple if there exists a finite sequence $(t_0, ...t_n)$ with $a = t_0 \leq ... \leq t_n = b$ and flows $f_i : [t_{i-1}, t_i] \to \mathcal{A}$, $i = 1, ..., n$ such that for $s \in [t_{i-1}, t_i]$, $f(s) = f_i(s) \cup \bigcup_{j=1}^{i-1} f_j(t_j)$.*

Any process $X$ indexed by $\mathcal{A}$ can be projected by a simple flow $f$ onto a process indexed by a subset of $\mathbf{R}$:

**Definition 6.2** *Let $X$ be an $\mathcal{A}$-indexed process and $f$ a simple flow on $S = [a, b]$. Then the $S$-indexed process $X^f$ is defined as follows:*

$$X^f(s) := X_{f(s)}, \ \forall s \in S.$$

*$X^f$ is called the projection of $X$ along $f$.*

In the case that $X$ can be extended to an additive process on $\tilde{\mathcal{A}}(u)$, $X$ can be projected by any arbitrary flow according to the preceding definition.

The following lemma shows the importance of the concept of flows for set-indexed processes.

Let $S(\mathcal{A})$ denote the class of simple continuous flows defined on $[0, 1]$.

**Lemma 6.3** (**[6]**) *The finite dimensional distributions of an (additive) $\mathcal{A}$-indexed process $X$ determine and are determined by the finite dimensional distributions of the class $\{X^f : f \in S(\mathcal{A})\}$.*

In this section, we study the set-indexed fractional Brownian motion on flows.

## 6.2 Levy fractional Brownian motion and fractional Brownian sheet on flows

As a preliminary, let us study the classical cases of Levy fractional Brownian motion and fractional Brownian sheet.
Let us consider $\mathcal{T} = \mathbf{R}_+^N$ and $\mathcal{A} = \left\{[0, t];\ t \in \mathbf{R}_+^N\right\}$. We can associate to any flow $f$, an increasing function $\tilde{f} : [0, 1] \to \mathbf{R}_+^N$ such that

$$\forall t \in [0, 1]; \quad f(t) = [0, \tilde{f}(t)]$$

- If $X$ is a Levy fBm,

$$\forall s, t \in [0, 1]; \quad E\left[X_{\tilde{f}(t)} - X_{\tilde{f}(s)}\right]^2 = \|\tilde{f}(t) - \tilde{f}(s)\|^{2H}$$

  Then, if $\tilde{f}(t) = t.\alpha$, where $\alpha \in \mathbf{R}_+^N$, $X^{\tilde{f}}$ is a classical fractional Brownian motion, otherwise it is not.

- If $X$ is a fBs,

$$\forall s, t \in [0, 1]; \quad E\left[X_{\tilde{f}(s)}.X_{\tilde{f}(t)}\right] = \prod_{i=1}^{N} \frac{1}{2}\left[\tilde{f}_i(s)^{2H} + \tilde{f}_i(t)^{2H} - |\tilde{f}_i(t) - \tilde{f}_i(s)|^{2H}\right]$$

Then, if the function $\tilde{f}$ is a line parallel to one axis of $\mathbf{R}_+^N$, the process $X^{\tilde{f}}$ is a fractional Brownian motion. However, if $\tilde{f}(t) = t.\alpha$, where $\alpha \in \mathbf{R}_+^N$,

$$\forall s,t \in [0,1]; \quad E\left[X_{\tilde{f}(s)}.X_{\tilde{f}(t)}\right] = \left[s^{2H} + t^{2H} - |t-s|^{2H}\right]^N \prod_{i=1}^{N} \frac{\alpha_i^{2H}}{2}$$

which is not a fBm.

In the two cases of the classical multiparameter extensions of the fractional Brownian motion, we saw that the projection of the process along a flow, is not in general a real-indexed fBm.

## 6.3 SifBm on flows is a standard fBm

Our definition for a set-indexed fractional Brownian motion is also justified by the following proposition.

**Proposition 6.4** *Let $X$ be a set-indexed fractional Brownian motion, and $f$ be a flow on $[0,1]$. Then the process $X^f = \left\{X_{f(t)};\ t \in [0,1]\right\}$ is a time-changed fractional Brownian motion.*

**Proof** The process $X^f$ is clearly a mean zero Gaussian process indexed by $[0,1]$. Moreover, its covariance function can be computed

$$E\left[X_{f(s)}X_{f(t)}\right] = \frac{1}{2}\left\{m[f(s)]^{2H} + m[f(t)]^{2H} - m[f(s) \bigtriangleup f(t)]^{2H}\right\}$$

For all $s \leq t$, we have $f(s) \subseteq f(t)$ and then

$$\begin{aligned} E\left[X_{f(s)}X_{f(t)}\right] &= \frac{1}{2}\left\{m[f(s)]^{2H} + m[f(t)]^{2H} - m[f(t) \setminus f(s)]^{2H}\right\} \\ &= \frac{1}{2}\left\{m[f(s)]^{2H} + m[f(t)]^{2H} - (m[f(t)] - m[f(s)])^{2H}\right\} \end{aligned}$$

The function $\theta : [0,1] \to \mathbf{R}_+$ such that for all $t \in [0,1]$, $\theta(t) = m[f(t)]$ is clearly increasing. Thus it defines a time change and we have

$$\forall s,t \in [0,1]; \quad E\left[X_{f(s)}X_{f(t)}\right] = \frac{1}{2}\left\{\theta(s)^{2H} + \theta(t)^{2H} - |\theta(t) - \theta(s)|^{2H}\right\} \quad (23)$$

Then $\left\{X_{f\circ\theta^{-1}(t)};\ t \in \mathbf{R}_+\right\}$ is a classical fractional Brownian motion. □

Proposition 6.4 allows to identify the self-similarity index of the SifBm, as the Hölder exponent of the projection along any flow.

Let us recall the definition of the two classical Hölder exponents of a stochastic process $X$ at $t_0 \in \mathbf{R}_+$ :

- the pointwise Hölder exponent

$$\alpha_X(t_0) = \sup\left\{\alpha : \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\rho^\alpha} < \infty\right\}$$

- the local Hölder exponent

$$\tilde{\alpha}_X(t_0) = \sup \left\{ \alpha : \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{|t-s|^\alpha} < \infty \right\}$$

**Corollary 6.5** *Let $X$ be a set-indexed fractional Brownian motion with self-similarity index $H$. The pointwise and local Hölder exponents of the projection $X^f$ along any flows $f$ at $t_0 \in [0,1]$, satisfy almost surely*

$$\alpha_{X^f}(t_0) = \begin{cases} \alpha_\theta(t_0).H & \text{if } \alpha_\theta(t_0) < 1 \\ H & \text{otherwise} \end{cases}$$

$$\tilde{\alpha}_{X^f}(t_0) = \begin{cases} \tilde{\alpha}_\theta(t_0).H & \text{if } \tilde{\alpha}_\theta(t_0) < 1 \\ H & \text{otherwise} \end{cases}$$

*where $\theta$ is the real function such that $\theta(t) = m[f(t)]$ ($\forall t \in [0,1]$), and $\alpha_\theta(t_0)$ (resp. $\tilde{\alpha}_\theta(t_0)$) is the pointwise (resp. local) Hölder exponent of $\theta$ at $t_0$.*

**Proof** Let $f$ be a flow and $X^f$ the projection of $X$ along $f$. By (23), we have

$$\forall s,t \in [0,1]; \quad E\left[X_t^f - X_s^f\right]^2 = |\theta(t) - \theta(s)|^{2H} \tag{24}$$

- If $\theta$ is differentiable on $[0,1]$, for all $t_0 \in [0,1]$ and $\rho > 0$,

  $$\forall s,t \in B(t_0,\rho); \quad |\theta(t) - \theta(s)| \sim K.|t-s|$$

  as $\rho$ tends to 0. Then,

  $$\forall s,t \in B(t_0,\rho); \quad E\left[X_t^f - X_s^f\right]^2 \sim K.|t-s|^{2H} \tag{25}$$

  In [5], we see that equation (25) implies

  $$P\{\forall t_0 \in [0,1];\ \alpha_{X^f}(t_0) = \tilde{\alpha}_{X^f}(t_0) = H\} = 1$$

- If $\theta$ is not differentiable in $t_0 \in [0,1]$ (i.e. if $\tilde{\alpha}_\theta(t_0) < 1$),

  $$\forall \alpha < \tilde{\alpha}_\theta(t_0); \quad \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t^f - X_s^f\right]^2}{|t-s|^{2\alpha.H}} = 0$$

  and

  $$\forall \alpha > \tilde{\alpha}_\theta(t_0); \quad \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t^f - X_s^f\right]^2}{|t-s|^{2\alpha.H}} = +\infty$$

  then, $\tilde{\alpha}_{X^f}(t_0) = \tilde{\alpha}_\theta(t_0).H$ almost surely (see [5]).
  In the same way, we get $\alpha_{X^f}(t_0) = \alpha_\theta(t_0).H$ almost surely.

□

**Remark 6.6** *As a set-indexed process on a flow only depends on its covariance between subsets $U$ and $V$ such that $U \subset V$, the result stated in proposition 6.4 still holds for the set-indexed mean-zero Gaussian process defined by (18). More generally, it holds for all process which is self-similar and $\mathcal{C}_0$-increment stationary. Therefore this result gives a supplementary legitimity that such a process could be called a set-indexed fractional Brownian motion.*

# 7 Concluding remarks

The preceeding discussions permit the following general definition

**Definition 7.1** *A set-indexed Gaussian process which is self-similar of index $H \in (0,1)$ and $\mathcal{C}_0$-increment stationary, is called general set-indexed fractional Brownian motion.*

**Corollary 7.2** *Let $X = \{X_U;\ U \in \mathcal{A}\}$ be a general set-indexed fractional Brownian motion. Then,*

1. *For all $H \in (0,1)$, such a process exists (Definition 2.11),*
2. *for any flow $f$, $X^f$ is a time-changed real-indexed fractional Brownian motion,*
3. *the covariance function between two subsets $U$ and $V$ such that $U \subseteq V$ is*

$$E\left[X_U X_V\right] = K\ \left[m(U)^{2H} + m(V)^{2H} - m(V \setminus U)^{2H}\right].$$


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# Chapitre IV

## Fine analysis of the regularity of Gaussian processes : Stochastic 2-microlocal analysis

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# Fine analysis of the regularity of Gaussian processes: Stochastic 2-microlocal analysis

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**Abstract**

A lot is known about the Hölder regularity of Gaussian processes. Recently, a finer analysis of the local regularity of functions, termed *2-microlocal analysis*, has been introduced in a deterministic frame: Through the computation of the so-called 2-microlocal frontier, it allows in particular to predict the evolution of regularity under the action of (pseudo-) differential operators. In this work, we apply 2-microlocal analysis to the study of Gaussian processes. We show that the study of the incremental covariance yields the almost sure value of the 2-microlocal frontier. As an application, we obtain new and refined regularity properties of fractional Brownian motion, multifractional Brownian motion, and stochastic generalized Weierstrass functions. Finally, we give a preliminary result concerning solutions of simple SDE.

*AMS classification*: 62 G 05, 60 G 15, 60 G 17, 60 G 18.
*Keywords*: 2-microlocal analysis, (multi)fractional Brownian motion, Gaussian processes, Hölder regularity, multi-parameter processes.

## 1 2-microlocal analysis

2-microlocal analysis, introduced by J.M. Bony ([8]), provides a tool that allows to predict the evolution of the local regularity of a function (or a process) under the action of (pseudo-)differential operators. To be more precise, let $f^{(\varepsilon)}$ denote the fractional integral (when $\varepsilon < 0$) or fractional derivative (when $\varepsilon > 0$) of the real function $f$. The pointwise Hölder exponent of $f^{(\varepsilon)}$ at $t$ is denoted $\alpha_{f^{(\varepsilon)}}(t)$ (see definition 2.2). In several applications (*e.g.* PDE, signal or image processing), one needs to have access to the function $\mathcal{H}_t : \varepsilon \mapsto \alpha_{f^{(\varepsilon)}}(t)$. Knowledge of $\mathcal{H}$ allows to answer questions such as: How much does one (locally) regularize the process $f$ by integrating it ? The problem comes from the well-known fact that the pointwise Hölder exponent is not stable under integro-differentiation: While it is true in simple situations that $\alpha_{f^{(n)}}(t) = \alpha_f(t) - n$, in general, one can only ensure that $\alpha_{f^{(n)}}(t) \leq \alpha_f(t) - n$. 2-microlocal analysis provides a way to assess the evolution of $\alpha_f$ through the use of a fine scale of functional spaces. These *2-microlocal spaces*, denoted $C^{s,s'}$, generalize the classical Hölder spaces in a way we describe now.

Since $\mathcal{H}$ cannot be deduced from the sole knowledge of $\alpha_f$[1], predicting changes in the regularity of a process under integro-differentiation basically requires recording the whole function $\varepsilon \mapsto \alpha_{f^{(\varepsilon)}}(t)$. 2-microlocal analysis does this in a clever way: It associates to any given point $t$ a curve in a abstract space, its *2-microlocal frontier*, whose slope is the rate of increase of $\varepsilon \mapsto \alpha_{f^{(\varepsilon)}}(t)$. The 2-microlocal frontier may be estimated through a fine analysis of the local regularity of $f$ around $t$. This analysis can be conducted in the Fourier ([8]), wavelet ([13]) or time ([18, 22]) domains. We shall use in this work the time-domain characterization of 2-microlocal spaces:

**Definition 1.1** *(Time domain definition of 2-microlocal spaces) Let $x_0 \in \boldsymbol{R}$, and $s, s'$ be two real numbers satisfying $s + s' > 0$, $s + s' \notin \boldsymbol{N}$, and $s' < 0$ (and thus $s \geq 0$). Let $m = [s + s']$.*

*A function $f : \boldsymbol{R} \to \boldsymbol{R}$ belongs to $C^{s,s'}_{x_0}$ if and only if its $m^{th}$ derivative exists around $x_0$, and if there exist $0 < \delta < 1/4$, a polynomial $P$ of degree not larger than $[s] - m$, and a constant $C$, that verify*

$$\left| \frac{\partial^m f(x) - P(x)}{|x - x_0|^{[s]-m}} - \frac{\partial^m f(y) - P(y)}{|y - x_0|^{[s]-m}} \right| \leq C|x-y|^{s+s'-m}(|x-y|+|x-x_0|)^{-s'-[s]+m}$$

*for all $x, y$ such that $0 < |x - x_0| < \delta$, $0 < |y - x_0| < \delta$.*

In the sequel, we shall restrict to the case where $(s, s')$ verify $0 < s + s' < 1, s < 1, s' < 0$. This corresponds to the situation where $f$ is continuous but not differentiable. This restriction allows to avoid certain technicalities in the analysis. We hope to treat the general case in a future work. When $(s, s')$ satisfy the above inequalities, the definition of 2-microlocal spaces reduces to:

**Definition 1.2** *(Time domain definition of 2-microlocal spaces, simple case) Let $D = \{(s, s') : 0 < s + s' < 1, s < 1, s' < 0\}$. A function $f : \boldsymbol{R} \to \boldsymbol{R}$ belongs to $C^{s,s'}_{x_0}$, with $(s, s') \in D$, if there exist a positive real $\delta$ and a constant $C$ such that for all $(x, y)$ with $0 < |x - x_0| < \delta$, $0 < |y - x_0| < \delta$,*

$$|f(x) - f(y)| \leq C|x - y|^{\sigma}(|x - x_0| + |x - y|)^{-s'}.$$

*where $\sigma = s + s'$.*

As we shall see below, performing the 2-microlocal analysis of a function $f$ at point $x$ allows to recover both the pointwise $\alpha$ and local $\tilde{\alpha}$ Hölder exponent of $f$ at $x$ (see definition 2.2). Let us first state some basic properties of 2-microlocal spaces ($C^{\alpha}_x$ denotes the usual pointwise Hölder space at $x$) :

[1]Because of the inequality $\alpha_{f^{(n)}} \leq \alpha_f - n$, $\mathcal{H}$ has to decrease faster than $\varepsilon \mapsto -\varepsilon$. One can show that, apart from this and a certain regularity property, there are no other constraints on the evolution of the pointwise Hölder exponent (see [11, 20]).

**Proposition 1.3** *$\forall x_0 \in \boldsymbol{R}$,*

- *$(t \le s$ and $t + t' \le s + s') \Rightarrow C_{x_0}^{s,s'} \subset C_{x_0}^{t,t'}$.*
- *$\forall s > 0$, $C_{x_0}^{s} \subset C_{x_0}^{s,-s}$.*
- *$\forall (s, s')$ with $s + s' > 0$, $C_{x_0}^{s,s'} \subset C_{x_0}^{s}$.*

Recall that the pointwise Hölder exponent of $f$ at $x$ is defined as the supremum of the $\alpha$ such that $f$ belongs to pointwise Hölder spaces $C_x^\alpha$. 2-microlocal spaces use two parameters $(s, s')$. The relevant notion generalizing the pointwise exponent is the *2-microlocal frontier*. In order to define this frontier, consider the *2-microlocal domain* of $f$ at $x_0$, i.e. the set $E(f, x_0) = \{(s, s') : f \in C_{x_0}^{s,s'}\}$. By Proposition 1.3, this set is a convex subset of the abstract plane $(s, s')$. The 2-microlocal frontier $\Gamma(f, x_0)$ is the convex curve in the $(s, s')$-plane defined by

$$\begin{aligned} \Gamma(f, x_0) : \quad & \mathbf{R} \to \mathbf{R} \\ & s' \mapsto s(s') = \sup\{r : f \in C_{x_0}^{r,s'}\} \end{aligned}$$

There are several ways to parameterize the 2-microlocal frontier, e.g. $s'(s)$, $s'(\sigma)$, ... (recall that $\sigma = s + s'$). Using $s$ as the free parameter does not allow to describe portions of the frontier where $s$ is constant (this may only happen on a half line $-\infty \le s' < s'_0$. An example is provided by the cusp $x \mapsto |x|^\gamma, \gamma > 0$, see figure 3). Likewise, the use of $\sigma$ as a free parameter is not possible on parts where $\sigma = s + s'$ is constant (this can only happen on a half line $s'_1 < s' < +\infty$. This occurs for the Weierstrass function $f(x)$, figure 3). Since no parts of the frontier may have $s'$ constant, this problem does not occur if one takes $s'$ as a free parameter. For various reasons, it is simpler to consider $\sigma(s')$ than $s(s')$, and this is the parameterization we shall mainly use in the following.

A simple consequence of Proposition 1.3 is the following:

**Proposition 1.4** *The 2-microlocal frontier of $f$ at any point $x_0$, seen as a function $s' \mapsto \sigma(s')$, verifies*

- *$\sigma(s')$ is a concave, non-decreasing function,*
- *$\sigma(s')$ has left and right derivatives always between 0 and 1.*

The next properties are fundamental (see Definition 2.2 for the definition of the local Hölder exponent $\tilde{\alpha}_f(x_0)$ of $f$ at $x_0$):

**Proposition 1.5** *(stability under fractional integro-differentiation) For any function $f : \boldsymbol{R} \to \boldsymbol{R}$, for all $(s, s') \in \boldsymbol{R}$, for all $x_0$ and for all $\varepsilon$*

$$f \in C_{x_0}^{s,s'} \iff f^{(\varepsilon)} \in C_{x_0}^{s-\varepsilon,s'}.$$

**Proposition 1.6** *(pointwise Hölder exponent) Assume $f \in C^\eta(\boldsymbol{R})$ for some $\eta > 0$. Then $\alpha_f(x_0) = -\inf\{s' : \sigma(s') \ge 0\}$, with the convention that $\alpha_f(x_0) = +\infty$ if $\sigma(s') \ge 0$ for all $s'$.*

**Proposition 1.7** *(local Hölder exponent) Assume $f \in C^\eta(\boldsymbol{R})$ for some $\eta > 0$. Then $\tilde{\alpha}_f(x_0) = \sigma(0)$.*

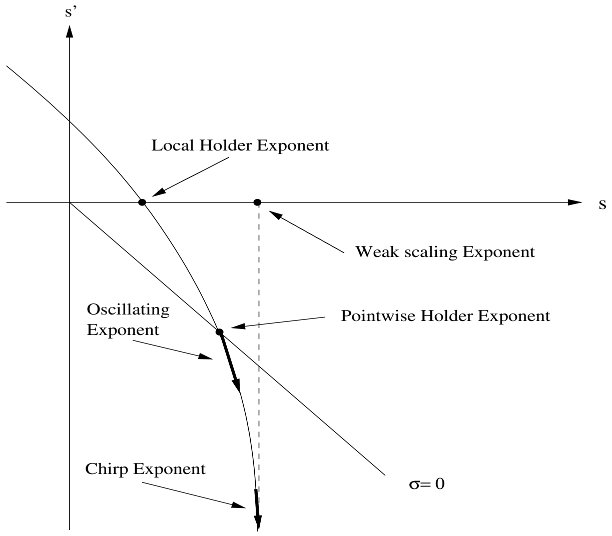


Figure 1: 2-microlocal frontier and exponents in the $(s, s')$-plane. See [11, 20] for the definitions of the chirp, oscillating and weak scaling exponents. These exponents are not used in this article.

The above propositions show that the 2-microlocal frontier contains the whole information pertaining to $\varepsilon \mapsto \alpha_{f^{(\varepsilon)}}(t)$. Indeed, fractional integro-differentiation of order $\varepsilon$ amounts to translating the 2-microlocal frontier by $-\varepsilon$ along the $s$ direction in the $(s, s')$ plane (proposition 1.5). The pointwise Hölder exponent of $f^{(\varepsilon)}$ is then given by the intersection of the translated frontier with the second bisector, provided $\varepsilon > \alpha_l$ (proposition 1.6). See figures 1 and 2 for a graphical illustration.

In order to obtain a more concrete understanding of 2-microlocal frontiers, let us consider some simple examples. The cusp function $x \mapsto |x|^\gamma$ has a trivial frontier at $x = 0$: It is vertical and passes through the point $(\gamma, 0)$. More interesting is the case of the chirp function $x \mapsto |x|^\gamma \sin(\frac{1}{|x|^\beta})$ : Its frontier at 0 is the straight line defined by $\sigma(s') = \frac{1}{\beta+1}s' + \frac{\gamma}{\beta+1}$.

Finally, let us consider the Weierstrass function $W_s(x) = \sum_{n=1}^{+\infty} \lambda^{-nh} \sin(\lambda^n x)$, where $\lambda > 2$, $0 < h < 1$. Its frontier is the same at all points $x$: In the $(s, s')$-plane, the frontier is vertical with $s = h$ for $s' \leq 0$, and it is parallel to the second bisector, i.e. $s' = h - s$, for $s' \geq 0$.

The three 2-microlocal frontiers are illustrated on figure 3. This ends our recalls on 2-microlocal analysis. See [8, 13, 19] for more detailed expositions.

## 2 2-microlocal analysis of Gaussian processes

In the remaining of this article, we shall perform the 2-microlocal analysis of Gaussian processes. We start by transposing the notion described in the previous section in a stochastic frame, and by defining some quantities that will prove useful for computing the almost-sure frontier of our processes.

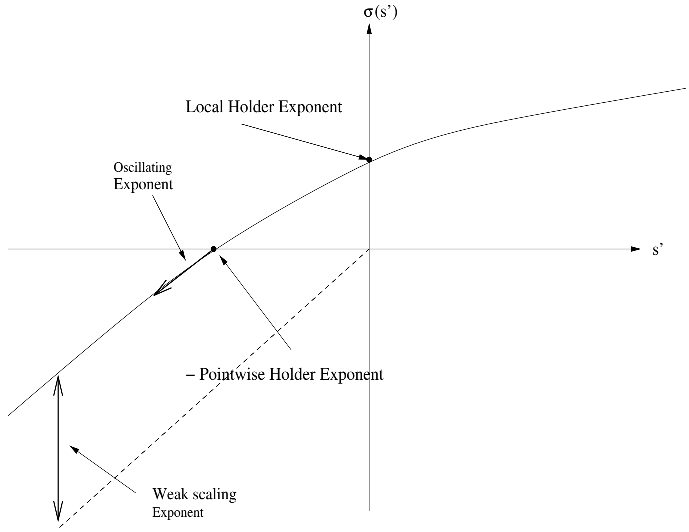


Figure 2: 2-microlocal frontier and exponents in the $(s', \sigma)$-plane.

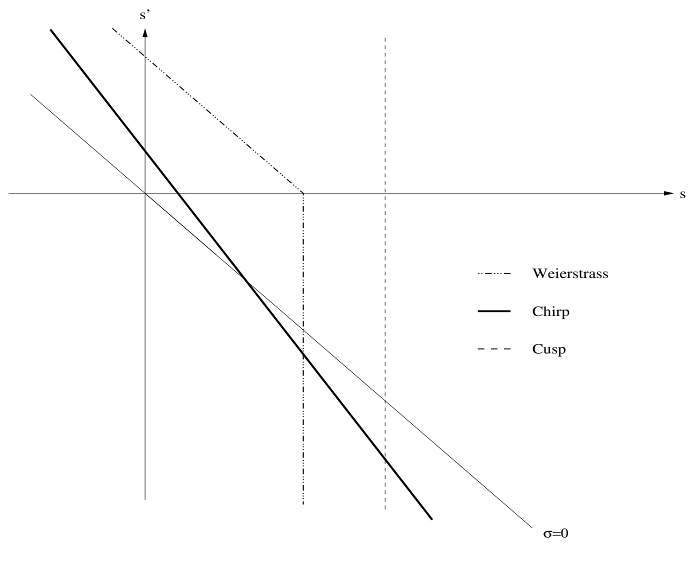


Figure 3: Examples of 2-microlocal frontiers: Cusp function, Chirp function, and a Weierstrass function.

## 2.1 Stochastic 2-microlocal analysis

Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a continuous Gaussian process. For each $t_0 \in \mathbf{R}_+^N$, let us define the 2-*microlocal frontier* of $X$ at $t_0$ as the random function $s' \mapsto \boldsymbol{\sigma}_{t_0}(s')$ defined for $s' \in (-\infty; 0)$,

$$\boldsymbol{\sigma}_{t_0}(s') = \sup\left\{\sigma;\ \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^{\sigma}\rho^{-s'}} < \infty\right\} \tag{1}$$

Each couple $(s'; \boldsymbol{\sigma}_{t_0}(s'))$ could be called a 2-*microlocal exponent* of $X$ at $t_0$. For a given realization $\omega$, $X(\omega) \in C_{x_0}^{s,s'}$ whenever $s < \boldsymbol{\sigma}_{t_0}(s') - s'$. In the same way as it is of no real interest to study, e.g., the continuity of a particular path $X(\omega)$, a statement like: $X(\omega) \in C_{x_0}^{s,s'}$ does not hold any relevant information. Rather, we seek to obtain results such as: $X \in C_{x_0}^{s,s'}$ almost surely. In that view, we shall study the set of couples $(s', \sigma)$ such that

$$\limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\sigma}\rho^{-2s'}} < +\infty \tag{2}$$

This kind of quantity is of course classical in the analysis of Gaussian processes (see, e.g. [9]). It leads naturally to define *deterministic 2-microlocal spaces* as follows:

**Definition 2.1** *A Gaussian process $X$ is said to belong to $\mathbb{C}_{t_0}^{s,s'}$ for a fixed $t_0 \in \boldsymbol{R}_+^N$ and some $s, s'$ such that*

$$\begin{cases} 0 < s + s' < 1 \\ s < 1 \\ s' < 0 \end{cases} \tag{3}$$

*if condition (2) is satisfied for $\sigma = s + s'$.*

In the sequel, we shall always assume that $\boldsymbol{\sigma}_{t_0}$ intersects the region defined by conditions (3). Recall that this is equivalent to assuming that $X$ is continuous but not differentiable at $t_0$.

Recall the definitions of the pointwise and local Hölder exponents of $X$ at $t_0$:

**Definition 2.2** *The pointwise and local Hölder exponents of $X$ at $t_0$ are defined as:*

$$\boldsymbol{\alpha}(t_0) = \sup\left\{\alpha; \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|X_t - X_s|}{\rho^{\alpha}} < \infty\right\}$$

$$\tilde{\boldsymbol{\alpha}}(t_0) = \sup\left\{\alpha; \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^{\alpha}} < \infty\right\}$$

It is easily seen that for all $t_0$ and all $s' < 0$, we have

$$\tilde{\boldsymbol{\alpha}}(t_0) \le \boldsymbol{\sigma}_{t_0}(s') - s' \le \boldsymbol{\alpha}(t_0) \tag{4}$$

To show this inequality, proceed as follows:

- For all $0 < \sigma < \boldsymbol{\sigma}_{t_0}(s')$,

$$\frac{|X_t - X_s|}{\rho^{\sigma - s'}} = \frac{|X_t - X_s|}{\|t-s\|^{\sigma} \rho^{-s'}} \left( \frac{\|t-s\|}{\rho} \right)^{\sigma}$$

which gives $\sigma - s' \leq \boldsymbol{\alpha}(t_0)$. As a consequence, $\boldsymbol{\sigma}_{t_0}(s') - s' \leq \boldsymbol{\alpha}(t_0)$.

- For all $\alpha < \tilde{\boldsymbol{\alpha}}(t_0)$,

$$\frac{|X_t - X_s|}{\|t-s\|^{\alpha + s'} \rho^{-s'}} = \frac{|X_t - X_s|}{\|t-s\|^{\alpha}} \left( \frac{\|t-s\|}{\rho} \right)^{-s'}$$

which gives $\alpha + s' \leq \boldsymbol{\sigma}_{t_0}(s')$. As a consequence, $\tilde{\boldsymbol{\alpha}}(t_0) \leq \boldsymbol{\sigma}_{t_0}(s') - s'$.

## 2.2 Where is, almost surely, the 2-microlocal frontier ?

In this section, we show that, not surprisingly, the 2-microlocal frontier of the paths of a Gaussian process can be evaluated by studying its incremental covariance. The proofs rely on the computation of almost sure lower and upper bounds for the frontier, which are developed in section 4.

As a counterpart to the random Hölder exponents and frontier $\boldsymbol{\sigma}_{t_0}(s')$, let us introduce the *deterministic local Hölder exponent*

$$\tilde{\mathbb{\alpha}}(t_0) = \sup \left\{ \alpha;\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\alpha}} < \infty \right\} \tag{5}$$

and the *deterministic* 2-*microlocal frontier* $s' \mapsto \mathbb{\sigma}_{t_0}(s')$:

$$\begin{aligned} \mathbb{\sigma}_{t_0}(s') &= \sup \left\{ \sigma;\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\sigma} \rho^{-2s'}} < \infty \right\} \\ &= \sup \left\{ \sigma;\ X \in \mathbb{C}_{t_0}^{\sigma - s', s'} \right\} \end{aligned} \tag{6}$$

The same proof as in the frame of deterministic functions allows to show that the deterministic 2-microlocal frontier $s' \mapsto \mathbb{\sigma}_{t_0}(s')$ is concave and thus continuous on $(-1, 0)$.

### 2.2.1 Pointwise almost sure 2-microlocal frontier

Proposition 4.1 in section 4 shows that for all $s'$, almost surely, $\mathbb{\sigma}_{t_0}(s') \leq \boldsymbol{\sigma}_{t_0}(s')$. Conversely, according to lemma 4.6, for all $s'$, almost surely, $\boldsymbol{\sigma}_{t_0}(s') \leq \mathbb{\sigma}_{t_0}(s')$. Using additionally the continuity of the frontier, one may thus state

**Theorem 2.3** *Let* $X = \left\{ X_t;\ t \in \boldsymbol{R}_+^N \right\}$ *be a continuous Gaussian process. For any* $t_0 \in \boldsymbol{R}_+^N$, *the* 2-*microlocal frontier of* $X$ *at* $t_0$ *is almost surely equal to the function* $s' \mapsto \mathbb{\sigma}_{t_0}(s')$.

**Corollary 2.4** *For any* $t_0 \in \boldsymbol{R}_+^N$, *the pointwise Hölder exponent of* $X$ *at* $t_0$ *is almost surely equal to* $-\inf\{s' : \mathbb{\sigma}_{t_0}(s') \geq 0\}$, *provided* $\mathbb{\sigma}_{t_0}(0) > 0$.

**Corollary 2.5** *For any* $t_0 \in \boldsymbol{R}_+^N$, *the local Hölder exponent of* $X$ *at* $t_0$ *is almost surely equal to* $\mathbb{\sigma}_{t_0}(0)$, *provided* $\mathbb{\sigma}_{t_0}(0) > 0$.

### 2.2.2 Uniform almost sure result on $\mathbf{R}_+^N$

Proposition 4.4 and theorem 4.9 in section 4 provide some almost sure results about the 2-microlocal frontier and the local Hölder exponent, uniformly in $t_0 \in \mathbf{R}_+^N$.

**Theorem 2.6** *Let $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ be a continuous Gaussian process. Assume that the function $t_0 \mapsto \liminf_{u \to t_0} \tilde{\mathbb{\alpha}}(u)$ is positive. Then, almost surely*

$$\forall t_0 \in \boldsymbol{R}_+^N;\ \liminf_{u \to t_0} \tilde{\mathbb{\alpha}}(u) \leq \tilde{\boldsymbol{\alpha}}(t_0) \leq \limsup_{u \to t_0} \tilde{\mathbb{\alpha}}(u) \tag{7}$$

**Proof** By definition of $\tilde{\mathbb{\alpha}}(t_0)$, for all $\epsilon > 0$, and all $t_0 \in \mathbf{R}_+^N$, there exist $C_0 > 0$ and $\rho_0 > 0$ such that

$$\forall s, t \in B(t_0, \rho_0);\ E\left[X_t - X_s\right]^2 \leq C_0 \|t - s\|^{2(\tilde{\mathbb{\alpha}}(t_0) - \epsilon)}$$

Then, proposition 4.4 implies that, almost surely,

$$\forall t_0 \in \mathbf{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t_0) \geq \liminf_{u \to t_0} \tilde{\mathbb{\alpha}}(u) - \epsilon$$

And, taking $\epsilon \in \mathbf{Q}_+$,

$$\forall t_0 \in \mathbf{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t_0) \geq \liminf_{u \to t_0} \tilde{\mathbb{\alpha}}(u)$$

Conversely, using theorem 4.9 with $s' = 0$, for all $\epsilon > 0$, we have almost surely

$$\forall t_0 \in \mathbf{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t_0) \leq \limsup_{u \to t_0} \tilde{\mathbb{\alpha}}(u) + \epsilon$$

Taking $\epsilon \in \mathbf{Q}_+$, we get

$$\forall t_0 \in \mathbf{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t_0) \leq \limsup_{u \to t_0} \tilde{\mathbb{\alpha}}(u)$$

□

**Corollary 2.7** *Let $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ be a continuous Gaussian process. Assume that the function $t_0 \mapsto \tilde{\mathbb{\alpha}}(t_0)$ is continuous and positive. Then, almost surely*

$$\forall t_0 \in \boldsymbol{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t_0) = \tilde{\mathbb{\alpha}}(t_0) \tag{8}$$

By remark 2.2, theorems 4.9 and 2.6 imply

**Corollary 2.8** *Let $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ be a Gaussian process. Assume that the function $t_0 \mapsto \tilde{\mathbb{\alpha}}(t_0)$ is continuous and positive. Then, almost surely*

$$\forall t_0 \in \boldsymbol{R}_+^N, \forall s' < 0;\ \tilde{\mathbb{\alpha}}(t_0) + s' \leq \boldsymbol{\sigma}_{t_0}(s') \leq \limsup_{u \to t_0} \mathbb{\sigma}_u(s') \tag{9}$$

Theorem 2.6 and corollary 2.8 only give bounds for the uniform almost sure 2-microlocal frontier. However, in specific cases, we are able to obtain a sharp result: this happens for instance for fractional Brownian motion and regular multifractional Brownian motion as we show below.

# 3 Applications to some well-known Gaussian processes

The results of the previous section can be used to compute the almost sure 2-microlocal frontier of some well-known Gaussian processes such as (multi)fractional Brownian motion and generalized Weierstrass function.

## 3.1 Fractional Brownian motion

Fractional Brownian motion (fBm) is one of the simplest processes whose regularity can be deeply studied (see [1] and [14] for the first results).
FBm is defined as the continuous Gaussian process $B^H = \left\{B_t^H; t \in \mathbf{R}_+\right\}$ such that for all $s, t \in \mathbf{R}_+$,

$$E\left[B_t^H - B_s^H\right]^2 = |t-s|^{2H} \tag{10}$$

where $H \in (0,1)$.
It is well-known that fBm is almost surely continuous but nowhere differentiable. As a consequence, its 2-microlocal frontier intersects the region defined by conditions (3). The results of paragraph 2.2.2 can then be applied to fBm. Theorem 2.6 directly yields the value of the almost sure local Hölder exponent uniformly on $\mathbf{R}_+$. The uniformity of (10) in the whole of $\mathbf{R}_+$, then allows to get the almost sure 2-microlocal frontier of fBm, uniformly in $\mathbf{R}_+$.

**Proposition 3.1** *Almost surely, the 2-microlocal frontier at any $t_0$ of the fractional Brownian motion in the region*

$$\begin{cases} 0 < \sigma < 1 \\ s' < 0 \\ \sigma - s' < 1 \end{cases}$$

*is equal to the line $\sigma = H + s'$.*

**Proof** For all $t_0 \in \mathbf{R}_+$ and all $\rho > 0$, we have

$$\forall s, t \in B(t_0, \rho);\ E\left[X_t - X_s\right]^2 = |t-s|^{2H}$$

Thus, it can be easily seen that

$$\forall t_0 \in \mathbf{R}_+;\ \tilde{\alpha}(t_0) = H \tag{11}$$

Conversely, for all $t_0 \in \mathbf{R}_+$, all $s' < 0$ and all sequence $(\rho_n)_{n\in\mathbf{N}}$ converging to 0, there exist two sequences $(s_n)_{n\in\mathbf{N}}$ and $(t_n)_{n\in\mathbf{N}}$ such that for all $n \in \mathbf{N}$, $s_n, t_n \in B(t_0, \rho_n)$ and $|t_n - s_n| = \rho_n$. Then, for all $n$, we have

$$\begin{aligned} \frac{E\left[X_{t_n} - X_{s_n}\right]^2}{|t_n - s_n|^{2(H+s')}\rho_n^{-2s'}} &= |t_n - s_n|^{-2s'}\rho_n^{2s'} \\ &= 1 \end{aligned}$$

which gives

$$\limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{|t-s|^{2(H+s')}\rho^{-2s'}} > 0$$

and then,

$$\forall t_0 \in \mathbf{R}_+, \forall s' < 0;\ \sigma_{t_0}(s') \leq H + s' \tag{12}$$

Therefore, using corollary 2.8, (11) and (12) imply almost surely

$$\forall t_0 \in \mathbf{R}_+, \forall s' < 0;\ \boldsymbol{\sigma}_{t_0}(s') = H + s' \tag{13}$$

□

As the pointwise (resp. local) Hölder exponent is the intersection of the 2-microlocal frontier with the axis $s' = 0$ (resp. the line $\sigma = 0$), one can state the following immediate consequence of proposition 3.1.

**Corollary 3.2** *The local and pointwise Hölder exponents satisfy almost surely*

$$\forall t_0 \in \boldsymbol{R}_+;\ \tilde{\boldsymbol{\alpha}}(t_0) = \boldsymbol{\alpha}(t_0) = H$$

## 3.2 Multifractional Brownian motion

As shown in corollary 3.2, the local regularity of fBm is constant along the paths. A natural extension of fBm is to substitute the constant parameter $H$, with a function $t \mapsto H(t)$ taking values in $(0, 1)$. This leads to multifractional Brownian motion (see [6], [21]). The mBm can be defined as the process $X = \{X_t; t \in \mathbf{R}_+\}$ such that

$$X_t = \int_{\mathbf{R}} \left[ |t-u|^{H(t)+\frac{1}{2}} - |u|^{H(t)+\frac{1}{2}} \right] .\mathbb{W}(du)$$

where $\mathbb{W}$ denotes the white noise of $\mathbf{R}$.
As a centered Gaussian process, the covariance structure of the mBm is not as simple as fBm's one (see [2]). The asymptotic behavior of the incremental covariance was stated in [12]. For all $a, b \in [0, 1]$, and all $t_0 \in [a, b]$, there exist positive constants $K(t_0)$ and $L(t_0)$ such that

$$\begin{aligned}\forall s, t \in B(t_0, \rho);\ E\left[X_t - X_s\right]^2 &= K(t_0)\|t-s\|^{2H(t)} + L(t_0)\left[H(t) - H(s)\right]^2 \\ &+ o_{a,b}\left(\|t-s\|^{2H(t)}\right) + o_{a,b}\left(H(t) - H(s)\right)^2 \end{aligned} \tag{14}$$

This approximation, together with the fact that mBm is continuous but not differentiable, allows us to compute the almost sure 2-microlocal frontier of the mBm at any point $t_0$. But on the contrary to fBm's case, (14) only gives local information about the covariance. As a consequence, one cannot state almost sure results uniformly in $t_0$ without further work.

### 3.2.1 Pointwise almost sure 2-microlocal frontier of the mBm

**Proposition 3.3** *The 2-microlocal frontier of the multifractional Brownian motion in the region*

$$\begin{cases} 0 < \sigma < 1 \\ s' < 0 \\ \sigma - s' < 1 \end{cases}$$

*is, at any fixed $t_0$, almost surely equal to the "minimum" of the 2-microlocal frontier of $H$ at $t_0$ and the line $\sigma = H(t_0) + s'$. More precisely, for all $t_0$, $\boldsymbol{\sigma}_{t_0}(s') = (H(t_0) + s') \wedge \beta_{t_0}(s')$ with probability one, where $\beta_{t_0}(s')$ denotes the 2-microlocal frontier of the deterministic function $H$ at $t_0$.*

**Proof** By definition, for each $s' \in (-\infty; 0)$,

$$\beta_{t_0}(s') = \sup\left\{\beta;\ \limsup_{\rho\to 0}\ \sup_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{||t-s||^{\beta}\rho^{-s'}} < \infty\right\} \tag{15}$$

We have to distinguish the 2 following cases:

- $H(t_0) + s' < \beta_{t_0}(s')$
  For all $\sigma < H(t_0) + s'$, there exists $\eta_0 > 0$ such that

$$\forall t \in B(t_0, \eta_0);\ \sigma < H(t) + s'$$

  Then, for all $0 < \rho < \eta_0$

$$\frac{||t-s||^{2H(t)}}{||t-s||^{2\sigma}\rho^{-2s'}} = \frac{||t-s||^{2(H(t)-\sigma)}}{\rho^{-2s'}} \le \frac{(2\rho)^{2(H(t)-\sigma)}}{\rho^{-2s'}} \to 0$$

  and

$$\frac{[H(t)-H(s)]^2}{||t-s||^{2\sigma}\rho^{-2s'}} \to 0$$

  Then by (14), we have $\sigma \le \varpi_{t_0}(s')$. This implies

$$H(t_0) + s' \le \varpi_{t_0}(s') \tag{16}$$

  For all $\sigma$ s.t. $H(t_0) + s' < \sigma < \beta_{t_0}(s')$, there exists $\eta_1 > 0$ such that

$$\forall t \in B(t_0, \eta_1);\ H(t) + s' < \sigma$$

  Let us consider $\rho_n = \frac{1}{n}$ and $s_n, t_n \in B(t_0, \rho_n)$ such that $|t_n - s_n| = \rho_n$. For $\frac{1}{n} \le \eta_1$, we have

$$\frac{||t_n-s_n||^{2H(t_n)}}{||t_n-s_n||^{2\sigma}\rho_n^{-2s'}} = \rho_n^{2(H(t_n)+s'-\sigma)} \to +\infty$$

  and

$$\frac{[H(t_n)-H(s_n)]^2}{||t_n-s_n||^{2\sigma}\rho_n^{-2s'}} \to 0$$

  Then from (14), we get $\sigma \ge \varpi_{t_0}(s')$. This implies

$$\varpi_{t_0}(s') \le H(t_0) + s' \tag{17}$$

- $\beta_{t_0}(s') < H(t_0) + s'$
  There exists $\eta_2 > 0$ such that

$$\forall t \in B(t_0, \eta_2);\ \beta_{t_0}(s') < H(t) + s'$$

  For all $\sigma < \beta_{t_0}(s')$, we have

$$\frac{||t-s||^{2H(t)}}{||t-s||^{2\sigma}\rho^{-2s'}} \to 0$$

and

$$\frac{[H(t)-H(s)]^2}{\|t-s\|^{2\sigma}\rho^{-2s'}} \to 0$$

Then, by (14), we have $\sigma \leq \boldsymbol{\sigma}_{t_0}(s')$. This implies

$$\beta_{t_0}(s') \leq \boldsymbol{\sigma}_{t_0}(s') \tag{18}$$

For all $\sigma$ s.t. $\beta_{t_0}(s') < \sigma < H(t_0) + s'$, there exist sequences

- $(\rho_n)_n$ of positive real numbers converging to 0,
- $(s_n)_n$ and $(t_n)_n$ s.t. $\forall n; s_n, t_n \in B(t_0, \rho_n)$

such that

$$\frac{[H(t_n)-H(s_n)]^2}{\|t_n-s_n\|^{2\sigma}\rho_n^{-2s'}} \to +\infty$$

Moreover, there exists $\eta_3 > 0$ such that

$$\forall t \in B(t_0, \eta_3);\ \sigma < H(t) + s'$$

Let $N_3 \in \mathbf{N}$ s.t. $\forall n \geq N_0;\ 0 < \rho_n < \eta_3$. For $n \geq N_3$, we have

$$\frac{\|t_n-s_n\|^{2H(t_n)}}{\|t_n-s_n\|^{2\sigma}\rho_n^{-2s'}} \to 0$$

Then, from (14), we get $\sigma \geq \boldsymbol{\sigma}_{t_0}(s')$. This implies

$$\boldsymbol{\sigma}_{t_0}(s') \leq \beta_{t_0}(s') \tag{19}$$

From (16), (17), (18) and (19), the result follows with theorem 2.3. □

**Corollary 3.4** *At any $t_0$, the pointwise and local Hölder exponents of the multifractional Brownian motion verify almost surely:*

$$\boldsymbol{\alpha}(t_0) = H(t_0) \wedge \beta(t_0)$$
$$\tilde{\boldsymbol{\alpha}}(t_0) = H(t_0) \wedge \tilde{\beta}(t_0)$$

*where $\beta(t_0)$ and $\tilde{\beta}(t_0)$ denote the pointwise and local Hölder exponents of $H$ at $t_0$.*

This result was already stated in [12].

#### 3.2.2 Uniform almost sure 2-microlocal frontier of mBm

Under some assumptions on the function $H$ or its regularity, uniform results hold. First, in the case where the local regularity of $H$ varies continuously, a direct application of theorem 2.6 yields the following statement:

**Proposition 3.5** *Let $X = \{X_t;\ t \in \boldsymbol{R}_+\}$ be a multifractional Brownian motion such that the function $t \mapsto \tilde{\beta}(t)$, where $\tilde{\beta}(t)$ is the local Hölder exponent of $H$ at $t$, is continuous on some open interval $I$. Then the local Hölder exponent of $X$ satisfy almost surely*

$$\forall t \in I:\ \tilde{\boldsymbol{\alpha}}(t) = H(t) \wedge \tilde{\beta}(t)$$

**Proof** Using the approximation (14), the deterministic Hölder exponent of $X$ at $t_0$ can be computed as in proposition 3.3

$$\tilde{\alpha}(t_0) = H(t_0) \wedge \tilde{\beta}(t_0)$$

The result follows from theorem 2.6. □

In the case of a regular mBm, i.e. when the values taken by the function $H$ are smaller than its regularity, a uniform result for the 2-microlocal frontier of the process holds as well:

**Theorem 3.6** *Let $X = \{X_t;\ t \in \boldsymbol{R}_+\}$ be a multifractional Brownian motion such that the function $H$ satisfy, for some open interval $I$,*

$$\forall t \in I : H(t) < \tilde{\beta}(t)$$

*where $\tilde{\beta}(t)$ is the local Hölder exponent of $H$ at $t$.*
*Then, almost surely, the 2-microlocal frontier at any $t_0 \in I$ of $X$ in the region*

$$\begin{cases} 0 < \sigma < 1 \\ s' < 0 \\ \sigma - s' < 1 \end{cases}$$

*is equal to the line $\sigma = H(t_0) + s'$.*
*In particular, almost surely, for all $t_0 \in I$, $\boldsymbol{\alpha}(t_0) = \tilde{\boldsymbol{\alpha}}(t_0) = H(t_0)$.*

**Proof** Under the assumptions of the theorem, for all $t_0 \in I$, (14) implies

$$\forall t_0 \in I;\ \tilde{\alpha}(t_0) = H(t_0) \tag{20}$$

Conversely, for all $t_0 \in I$, and all sequence $(\rho_n)_{n\in\mathbf{N}}$ converging to 0, there exist two sequences $(s_n)_{n\in\mathbf{N}}$ and $(t_n)_{n\in\mathbf{N}}$ such that for all $n \in \mathbf{N}$, $s_n, t_n \in B(t_0, \rho_n)$ and $|t_n - s_n| = \rho_n$.
Then, by (14), for all $\sigma$ s.t. $H(t_0) + s' < \sigma < \tilde{\beta}(t_0) + s'$, we have

$$\frac{E\left[X_{t_n} - X_{s_n}\right]^2}{|t_n - s_n|^{2\sigma} \rho_n^{-2s'}} \longrightarrow +\infty$$

as $n$ goes to $+\infty$, which gives

$$\limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{|t-s|^{2\sigma}\rho^{-2s'}} > 0$$

and thus

$$\forall t_0 \in I, \forall s' < 0;\ \sigma_{t_0}(s') \leq H(t_0) + s' \tag{21}$$

The result follows from (20), (21) and corollary 2.8. □

**Remark 3.7** *With global regularity conditions on the function $H$, one may obtain a uniform analog of Proposition 3.3. For instance, it is not hard to adapt the proofs above to show that if the inequality*

$$\limsup_{\rho\to 0}\ \sup_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{\|t-s\|^{\beta_{t_0}}\rho^{-s'}} < \infty$$

*is verified for all $t_0 \in I$, then, almost surely, for all $t_0 \in I$,*

$$\boldsymbol{\sigma}_{t_0}(s') = (H(t_0)+s') \wedge \beta_{t_0}(s')$$

*and*

$$\boldsymbol{\alpha}(t_0) = H(t_0) \wedge \beta(t_0)$$
$$\tilde{\boldsymbol{\alpha}}(t_0) = H(t_0) \wedge \tilde{\beta}(t_0)$$

*where $\beta(t_0)$ and $\tilde{\beta}(t_0)$ denote the pointwise and local Hölder exponents of $H$ at $t_0$.*

## 3.3 Generalized Weierstrass function

Let us recall the definition of the well-known Weierstrass function ([10]):

$$W_H(t) = \sum_{j=1}^{\infty} \lambda^{-jH} \sin \lambda^j t \tag{22}$$

where $\lambda > 2$ and $H \in (0,1)$.
Its 2-microlocal frontier is displayed on figure 3.
A stochastic version of the classical Weierstrass function has been studied in [5]. Consider $(Z_j)_{j\in\mathbf{N}}$ a sequence of $\mathcal{N}(0,1)$ i.i.d. random variables and define the *generalized Weierstrass function (GW)* as the Gaussian process $X = \{X_t; t \in \mathbf{R}_+\}$ such that

$$X_t = W_{H(t)}(t) = \sum_{j=1}^{\infty} \lambda^{-jH(t)} \sin \lambda^j t\ Z_j \tag{23}$$

where $t \mapsto H(t)$ takes values in $(0,1)$.
As in the mBm's case, the regularity of this process can be obtained by the computation of the incremental covariance. Indeed it is easy to show that $X$ is continuous but not differentiable.

### 3.3.1 Bound for the incremental covariance of GW

**Proposition 3.8** *Let $X = \{X_t;\ t \in \boldsymbol{R}_+\}$ be a generalized Weierstrass function. For all $0 < a < b$, there exists positive constants $K$ and $L$ such that*

$$\forall s,t \in [a,b];\ E\left[X_t - X_s\right]^2 \le K|t-s|^{H(s)+H(t)} + L\left(H(t)-H(s)\right)^2 \tag{24}$$

**Proof** From (23), we compute

$$\begin{aligned} E\left[X_t - X_s\right]^2 &= \sum_{j,k} \left(\lambda^{-jH(t)} \sin \lambda^j t - \lambda^{-jH(s)} \sin \lambda^j s\right) \\ &\quad \times \left(\lambda^{-kH(t)} \sin \lambda^k t - \lambda^{-kH(s)} \sin \lambda^k s\right) .E\left[Z_j Z_k\right] \\ &= \sum_{j} \left(\lambda^{-jH(t)} \sin \lambda^j t - \lambda^{-jH(s)} \sin \lambda^j s\right)^2 \qquad (25) \end{aligned}$$

Then, using the decomposition

$$\lambda^{-jH(t)} \sin \lambda^j t - \lambda^{-jH(s)} \sin \lambda^j s = \left(\lambda^{-jH(t)} - \lambda^{-jH(s)}\right) \sin \lambda^j t + \lambda^{-jH(s)} \left(\sin \lambda^j t - \sin \lambda^j s\right)$$

we get

$$\begin{aligned} E\left[X_t - X_s\right]^2 &\leq 2 \sum_{j} \left(\lambda^{-jH(t)} - \lambda^{-jH(s)}\right)^2 \sin^2 \lambda^j t \\ &\quad + 2 \sum_{j} \lambda^{-2jH(s)} \left(\sin \lambda^j t - \sin \lambda^j s\right)^2 \qquad (26) \end{aligned}$$

First, let us give an upper bound for the first term of (26).

$$\sum_{j=1}^{\infty} \left(\lambda^{-jH(t)} - \lambda^{-jH(s)}\right)^2 \sin^2 \lambda^j t \leq \sum_{j=1}^{\infty} \left(\lambda^{-jH(t)} - \lambda^{-jH(s)}\right)^2$$

By the finite increments theorem, there exists $\tau$ between $H(s)$ and $H(t)$ such that

$$\lambda^{-jH(t)} - \lambda^{-jH(s)} = -j\lambda^{-j\tau} \left(H(t) - H(s)\right) \log \lambda$$

therefore

$$\sum_{j=1}^{\infty} \left(\lambda^{-jH(t)} - \lambda^{-jH(s)}\right)^2 \sin^2 \lambda^j t \leq \left(H(t) - H(s)\right)^2 \log^2 \lambda . \sum_{j=1}^{\infty} j^2 \lambda^{-2a.j} \qquad (27)$$

To deal with the second term of (26), for all $s, t \in [a, b]$, we consider the integer $N$ such that $\lambda^{-(N+1)} \leq |t - s| \leq \lambda^{-N}$.

$$\sum_{j=1}^{\infty} \lambda^{-2jH(s)} \left(\sin \lambda^j t - \sin \lambda^j s\right)^2 \leq \sum_{j=1}^{N} \lambda^{-2jH(s)} \left(\sin \lambda^j t - \sin \lambda^j s\right)^2 + 4 \sum_{j=N+1}^{\infty} \lambda^{-2jH(s)} \qquad (28)$$

Then, using the inequality

$$\begin{aligned} \left(\sin \lambda^j t - \sin \lambda^j s\right)^2 &= 4 \sin^2 \lambda^j \frac{t-s}{2} \cos^2 \lambda^j \frac{t+s}{2} \\ &\leq \lambda^{2j} |t - s|^2 \leq \lambda^{2j} \lambda^{-2N} \end{aligned}$$

we get

$$\sum_{j=1}^{N} \lambda^{-2jH(s)} \left(\sin \lambda^j t - \sin \lambda^j s\right)^2 \leq \lambda^{-2N} \sum_{j=1}^{N} \lambda^{2j(1-H(s))}$$

$$\leq \lambda^{-2N} \lambda^{2(1-H(s))} \frac{\lambda^{2(N-1)(1-H(s))}}{\lambda^{2(1-H(s))} - 1}$$

$$\leq \frac{\lambda^{-2NH(s)}}{\lambda^{2(1-H(s))} - 1}$$

$$\leq \frac{\lambda^{2H(s)}}{\lambda^{2(1-H(s))} - 1}.|t-s|^{2H(s)}$$

Moreover, as

$$\sum_{j=N+1}^{\infty} \lambda^{-2jH(s)} = \frac{\lambda^{-2(N+1)H(s)}}{1-\lambda^{-2H(s)}}$$

$$\leq \frac{|t-s|^{2H(s)}}{1-\lambda^{-2H(s)}}$$

we have

$$\sum_{j=1}^{\infty} \lambda^{-2jH(s)} \left(\sin \lambda^j t - \sin \lambda^j s\right)^2 \leq \left(\frac{\lambda^{2H(s)}}{\lambda^{2(1-H(s))} - 1} + \frac{4}{1-\lambda^{-2H(s)}}\right).|t-s|^{2H(s)} \tag{29}$$

As

$$|t-s|^{2H(s)} - |t-s|^{H(s)+H(t)} = O_{a,b}\left(|t-s|^{H(s)+H(t)}(H(t)-H(s))\right) + O_{a,b}\left(H(t)-H(s)\right)^2$$

the result follows from (26), (27) and (29). □

To get a upper bound for the 2-microlocal frontier of the generalized Weierstrass function, we need the following statement

**Proposition 3.9** *Let $X = \{X_t;\ t \in \boldsymbol{R}_+\}$ be a generalized Weierstrass function. For all $t_0 \in \boldsymbol{R}_+$, there exists two sequences $(s_n)_{n\in\boldsymbol{N}}$ and $(t_n)_{n\in\boldsymbol{N}}$ converging to $t_0$ and positive constants $k_1$ and $l_1$ such that*

$$\forall n \in \boldsymbol{N};\ \left(E\left[X_{t_n} - X_{s_n}\right]^2\right)^{\frac{1}{2}} \geq k_1 |t_n - s_n|^{\frac{H(s_n)+H(t_n)}{2}} - l_1 \left|H(t_n) - H(s_n)\right|$$

*Moreover, if $H$ admits a positive local Hölder exponent at $t_0$, there exists positive constants $k_2$ and $l_2$ such that*

$$\forall n \in \boldsymbol{N};\ \left(E\left[X_{t_n} - X_{s_n}\right]^2\right)^{\frac{1}{2}} \geq -\left[k_2 \left|H(t_n) - H(s_n)\right| \log|t_n - s_n| + l_2\right] |t_n - s_n|^{\frac{H(s_n)+H(t_n)}{2}}$$

**Proof** From (25), for all $s, t \in \mathbf{R}_+$ and all $n \in \mathbf{N}$, we have

$$E\left[X_t - X_s\right]^2 \geq \left(\lambda^{-nH(t)} \sin \lambda^n t - \lambda^{-nH(s)} \sin \lambda^n s\right)^2$$

Using the decomposition

$$\lambda^{-nH(t)}\sin\lambda^n t - \lambda^{-nH(s)}\sin\lambda^n s = \left(\lambda^{-nH(t)} - \lambda^{-nH(s)}\right)\sin\lambda^n t + \lambda^{-nH(s)}\left(\sin\lambda^n t - \sin\lambda^n s\right)$$

and the triangular inequality, we get

$$E\left[X_t - X_s\right]^2 \geq \left(\left|\lambda^{-nH(t)} - \lambda^{-nH(s)}\right| . \left|\sin\lambda^n t\right| - \lambda^{-nH(s)}\left|\sin\lambda^n t - \sin\lambda^n s\right|\right)^2 \tag{30}$$

For all $t_0 \in \mathbf{R}_+$, there exists a sequence $(t_n)_{n\in\mathbf{N}}$ converging to $t_0$, and such that $|\sin\lambda^n t_n| > \frac{1}{2}$ for all $n \in \mathbf{N}$. For instance, let us start from a sequence $\left(\tilde{t}_n = t_0 + \frac{\pi}{\lambda^n}\right)_{n\in\mathbf{N}}$ converging to $t_0$, and set, for all $n$

$$t_n = \begin{cases} \tilde{t}_n \text{ if } |\sin\lambda^n t_n| > \frac{1}{2} \\ \tilde{t}_n + \frac{\pi}{2\lambda^n} \text{ otherwise} \end{cases}$$

Moreover, for all $t \in \mathbf{R}_+$ and all $n \in \mathbf{N}$, there exists $h_n$ such that $\lambda^{-(n+1)} \leq h_n \leq \lambda^{-n}$ and $|\sin\lambda^n(t+h_n) - \sin\lambda^n t| \geq \frac{1}{10}$.
As a consequence, setting $s_n = t_n + h_n$ for all $n$, we get a sequence $(s_n)_{n\in\mathbf{N}}$ converging to $t_0$, and such that

$$\forall n \in \mathbf{N};\ \left(E\left[X_{t_n} - X_{s_n}\right]^2\right)^{\frac{1}{2}} \geq \frac{1}{10}\lambda^{-nH(s_n)} - \left|\lambda^{-nH(t_n)} - \lambda^{-nH(s_n)}\right|$$

and

$$\forall n \in \mathbf{N};\ \left(E\left[X_{t_n} - X_{s_n}\right]^2\right)^{\frac{1}{2}} \geq \frac{1}{2}\left|\lambda^{-nH(t_n)} - \lambda^{-nH(s_n)}\right| - 2\lambda^{-nH(s_n)}$$

Let us consider the local Hölder exponent of $H$ at $t_0$

$$\tilde{\beta}(t_0) = \sup\left\{\beta;\ \limsup_{\rho\to 0}\ \sup_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{|t-s|^\beta} < +\infty\right\}$$

As $\tilde{\beta}(t_0) > 0$, we can choose $0 < \beta < \tilde{\beta}(t_0)$. This implies

$$\begin{aligned} n\left(H(t_n) - H(s_n)\right) &\sim -\left(H(t_n) - H(s_n)\right)\log|t_n - s_n| \\ &\sim -\underbrace{\frac{H(t_n) - H(s_n)}{|t_n - s_n|^\beta}}_{\to 0}\underbrace{|t_n - s_n|^\beta\log|t_n - s_n|}_{\to 0} \\ &\longrightarrow 0 \end{aligned}$$

Then, a Taylor's expansion gives

$$\lambda^{-nH(t_n)} - \lambda^{-nH(s_n)} = \left(H(t_n) - H(s_n)\right)n\log\lambda.\lambda^{-nH(s_n)} + O\left[\left(H(t_n) - H(s_n)\right)^2 n^2\lambda^{-nH(s_n)}\right]$$

Therefore, using $|t_n - s_n| \leq \lambda^{-n}$ and the boundedness of $n\lambda^{-nH(s_n)}$, there exists $l_1 > 0$ such that

$$\forall n \in \mathbf{N};\ \left(E\left[X_{t_n} - X_{s_n}\right]^2\right)^{\frac{1}{2}} \geq \frac{1}{10}|t_n - s_n|^{H(s_n)} - l_1\ |H(t_n) - H(s_n)|$$

and there exists $k_2 > 0$ such that

$$\forall n \in \mathbf{N};\ \left(E\left[X_{t_n} - X_{s_n}\right]^2\right)^{\frac{1}{2}} \geq k_2\left|H(t_n) - H(s_n)\right| \times |t_n - s_n|^{H(s_n)}\left(-\log|t_n - s_n|\right) - 2\left|t_n - s_n\right|^{H(s_n)}$$

We conclude in the same way as in the proof of proposition 3.8. □

### 3.3.2 Pointwise almost sure 2-microlocal frontier of GW

As in the mBm's case, propositions 3.8 and 3.9 give the almost sure 2-microlocal frontier of the generalized Weierstrass function, when $H$ is regular.

**Theorem 3.10** *Let $X = \{X_t;\ t \in \boldsymbol{R}_+\}$ be a generalized Weierstrass function such that the function $H$ satisfy, for some open interval $I$,*

$$\forall t \in I; H(t) < \tilde{\beta}(t)$$

*where $\tilde{\beta}(t)$ is the local Hölder exponent of $H$ at $t$.*
*Then, almost surely, the 2-microlocal frontier at any $t_0 \in I$ of $X$ in the region*

$$\begin{cases} 0 < \sigma < 1 \\ s' < 0 \\ \sigma - s' < 1 \end{cases}$$

*is equal to the line $\sigma = H(t_0) + s'$.*

**Proof** The proof is similar to the one of mBm's case. We sketch it below.
For each $s' \in (-\infty; 0)$, we introduce

$$\beta_{t_0}(s') = \sup\left\{\beta; \limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{\|t-s\|^{\beta}\rho^{-s'}} < \infty\right\} \tag{31}$$

From (4), for all $s' \in (-1, 0]$, we have $H(t_0) + s' < \beta_{t_0}(s')$. Then, for all $\sigma < H(t_0) + s'$, we have

$$\frac{\|t-s\|^{2H(t_0)}}{\|t-s\|^{2\sigma}\rho^{-2s'}} = \frac{\|t-s\|^{2(H(t_0)-\sigma)}}{\rho^{-2s'}} \leq \frac{(2\rho)^{2(H(t_0)-\sigma)}}{\rho^{-2s'}} \to 0$$

and

$$\frac{[H(t)-H(s)]^2}{\|t-s\|^{2\sigma}\rho^{-2s'}} \to 0$$

Thus, by proposition 3.8, we have $\sigma \leq \varpi_{t_0}(s')$. This implies

$$H(t_0) + s' \leq \varpi_{t_0}(s') \tag{32}$$

Conversely, for all $\sigma$ s.t. $H(t_0)+s' < \sigma < \beta_{t_0}(s')$, the two sequences $(s_n)_{n\in\mathbf{N}}$ and $(t_n)_{n\in\mathbf{N}}$ given by proposition 3.9 can be chosen such that for all $n \in \mathbf{N}$, $t_0 + \frac{\pi}{\lambda^n} \leq t_n \leq t_0 + \frac{3\pi}{2\lambda^n}$ and $\lambda^{-(n+1)} \leq s_n - t_n \leq \lambda^{-n}$.
As a consequence, we have

$$\forall n \in \mathbf{N}; \qquad t_0 \leq t_n \leq s_n \text{ and}$$
$$\lambda^{-n}\left(\frac{1}{\lambda}+\pi\right) \leq s_n - t_0 \leq \lambda^{-n}\left(1+\frac{3\pi}{2}\right)$$

Therefore, setting $\rho_n = s_n - t_0$, we have $s_n, t_n \in B(t_0, \rho_n)$ and

$$\frac{\|t_n-s_n\|^{2H(t_0)}}{\|t_n-s_n\|^{2\sigma}\rho_n^{-2s'}} \geq \frac{\lambda^{-2H(t_0)(n+1)}}{\lambda^{-2n\sigma}\lambda^{2ns'}\left(1+\frac{3\pi}{2}\right)^{-2s'}} = \lambda^{2n(H(t_0)+s'-\sigma)} \to +\infty$$

As, on the other hand,

$$\frac{[H(t_n) - H(s_n)]^2}{\|t_n - s_n\|^{2\sigma} \rho_n^{-2s'}} \to 0$$

from proposition 3.9, we get $\sigma \geq \mathbb{\sigma}_{t_0}(s')$. This implies

$$\mathbb{\sigma}_{t_0}(s') \leq H(t_0) + s' \tag{33}$$

From (32) and (33), we get

$$\begin{aligned} \forall t_0 \in I, \forall s' < 0; \quad & \mathbb{\sigma}_{t_0}(s') = H(t_0) + s' \\ & \mathbb{\alpha}(t_0) = H(t_0) \end{aligned}$$

Corollary 2.8 then gives the result. □

When $H(t) > \tilde{\beta}(t)$, we are not able to conclude in general. However, it should be possible to obtain a complete almost sure result if one uses the definition of the stochastic Weierstrass function used in [5] instead of (23). The trick consists in summing over a particular set of indices that grows sufficiently fast to infinity, rather than on the whole of $\mathbf{N}$. See [5] for details.

### 3.3.3 Uniform almost sure 2-microlocal frontier of GW on $\mathbf{R}_+^N$

**Proposition 3.11** *The 2-microlocal frontier at any $t_0$ of the generalized Weierstrass function in the region*

$$\begin{cases} 0 < \sigma < 1 \\ s' < 0 \\ \sigma - s' < 1 \end{cases}$$

*is, almost surely, "above the minimum" of the line $s' \mapsto H(t_0) + s'$ and the 2-microlocal frontier of $H$*

**Proof** We have to distinguish between the two following cases:

- If $H(t_0) + s' < \beta_{t_0}(s')$, theorem 3.10 gives the result.
- If $\beta_{t_0}(s') < H(t_0) + s'$
  For all $\sigma < \beta_{t_0}(s')$, we have

  $$\frac{\|t - s\|^{2H(t_0)}}{\|t - s\|^{2\sigma} \rho^{-2s'}} \to 0$$

  and

  $$\frac{[H(t) - H(s)]^2}{\|t - s\|^{2\sigma} \rho^{-2s'}} \to 0$$

  Then, by proposition 3.8 and proposition 4.1, we have $\sigma \leq \boldsymbol{\sigma}_{t_0}(s')$ almost surely. This implies almost surely

  $$\beta_{t_0}(s') \leq \boldsymbol{\sigma}_{t_0}(s') \tag{34}$$

□

**Remark 3.12** *Conversely, for all $\sigma$ s.t. $\beta_{t_0}(s') < \sigma < H(t_0) + s'$, there exist sequences $(\rho_n)_n$, $(s_n)_n$ and $(t_n)_n$ such that $\forall n; s_n, t_n \in B(t_0, \rho_n)$ and*

$$\frac{[H(t_n) - H(s_n)]^2}{\|t_n - s_n\|^{2\sigma} \rho_n^{-2s'}} \to +\infty$$

*Moreover, we have*

$$\frac{\|t_n - s_n\|^{2H(t_0)}}{\|t_n - s_n\|^{2\sigma} \rho_n^{-2s'}} \to 0$$

*But inequalities given by proposition 3.9 are not satisfied for the sequences $(s_n)_n$ and $(t_n)_n$. Then it cannot be used to get an almost sure upper bound for $\boldsymbol{\sigma}_{t_0}(s')$. Again, using the definition set in ([5]) instead of (23) should allow to conclude in general.*

## 3.4 Application to solutions of some SDE

Let us consider the classical stochastic differential equation

$$dX_t = \eta(t, X_t).dB_t + b(t, X_t).dt \tag{35}$$

where $\{B_t;\ t \in \mathbf{R}_+\}$ denotes the standard Brownian motion.
The goal of this section is to show that under some assumptions on $\eta$ and $b$, the almost sure 2-microlocal frontier of $X$ can be evaluated.

**Theorem 3.13** *Let $X$ be a stochastic process, solution of the equation*

$$dX_t = \eta(t).dB_t + b(t).dt \tag{36}$$

*Let $\Gamma_{t_0}$ (resp. $\Delta_{t_0}$) denote the 2-microlocal frontier of the function $t \mapsto \int_0^t \eta^2$ (resp. $t \mapsto \int_0^t b$) at $t_0 \in \boldsymbol{R}_+$. Assume that these frontiers intersect the region defined by conditions (3).*
*Then, the 2-microlocal frontier of $X$ at $t_0$ is almost surely equal to* $\inf(\mathcal{H}(\Gamma_{t_0}); \Delta_{t_0})$, *where $\mathcal{H}$ is the homothecy of center $O$ and ratio $\frac{1}{2}$.*

**Proof** If $X$ satisfy (36), we have

$$\forall s, t \in \mathbf{R}_+;\ X_t - X_s = \int_s^t \eta(u).dB_u + \int_s^t b(u).du$$

and by definition of the stochastic integral

$$\begin{aligned} \forall s, t \in \mathbf{R}_+;\ E\,[X_t - X_s]^2 &= \left| \int_s^t \eta^2(u).du \right| + \left( \int_s^t b(u).du \right)^2 \\ &= |\phi(t) - \phi(s)| + (\psi(t) - \psi(s))^2 \end{aligned}$$

where $\phi$ is the function $t \mapsto \int_0^t \eta^2(u).du$ and $\psi$ the function $t \mapsto \int_0^t b(u).du$. For all $t_0 \in \mathbf{R}_+$, let us consider $s' \mapsto \beta_{t_0}(s')$ and $s' \mapsto \gamma_{t_0}(s')$, the 2-microlocal functions of $\phi$ and $\psi$ at $t_0$. As for all $\rho > 0$ and all $\sigma > 0$ and $s' < 0$,

$$\forall s, t \in B(t_0, \rho); \; \frac{E\left[X_t - X_s\right]^2}{|t-s|^{2\sigma}\rho^{-2s'}} = \frac{|\phi(t) - \phi(s)|}{|t-s|^{2\sigma}\rho^{-2s'}} + \left(\frac{|\psi(t) - \psi(s)|}{|t-s|^{\sigma}\rho^{-s'}}\right)^2$$

The deterministic 2-microlocal function of $X$ at $t_0$ is therefore

$$\mathbb{\sigma}_{t_0}(s') = \frac{1}{2}\beta_{t_0}(2s') \wedge \gamma_{t_0}(s')$$

The result follows from theorem 2.3. □

# 4 Bound for the 2-microlocal frontier of Gaussian processes

## 4.1 Lower bound for the 2-microlocal frontier

Studying whether the paths of a stochastic process belong to some space $C_{t_0}^{s,s'}$, does only make sense if they are continuous. Then, in the following, we assume that the stochastic processes are continuous.

### 4.1.1 Pointwise almost sure result

**Proposition 4.1** *Consider a continuous Gaussian process $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$. Assume that for some $t_0$, there exists two constants $C > 0$ and $\rho_0 > 0$ such that*

$$\forall 0 < \rho < \rho_0, \forall s, t \in B(t_0, \rho), E\left[X_t - X_s\right]^2 \leq C\|t-s\|^{2\sigma}\rho^{-2s'} \tag{37}$$

*with $\sigma > 0$ and $s' < 0$.*
*Then, almost surely, the paths of the process $X$ belong to $C_{t_0}^{\tilde{\sigma}-s',s'}$, for all $\tilde{\sigma} < \sigma$.*
*In other words, if $X \in \mathbb{C}_{t_0}^{\sigma-s',s'}$ then almost surely $X \in C_{t_0}^{\tilde{\sigma}-s',s'}$, for all $\tilde{\sigma} < \sigma$.*

**Proof** Let $\epsilon > 0$ be such that $\tilde{\sigma} = \sigma - \epsilon > 0$. Let us take $\rho = 2^{-n}$ for $n \geq n_0 = -\log_2 \rho_0$ and set $D_n^m(t_0) = \left\{t_0 + k.2^{-(m+n)}; k \in \{0, \pm 1, \ldots, \pm 2^m\}^N\right\}$. Let us consider the event

$$\Omega_n^m = \left\{ \max_{\substack{k,l \in \{0,\ldots,\pm 2^m\}^N \\ \|k-l\|=1}} \left|X_{t_0 + k.2^{-(m+n)}} - X_{t_0 + l.2^{-(m+n)}}\right| > 2^{-\tilde{\sigma}(m+n)} 2^{s'n} \right\}$$

For all $p \in \mathbf{N}^*$, we have

$$\begin{aligned}
P\left\{\Omega_n^m\right\} &\leq \sum_{\substack{k,l\in\{0,\ldots,\pm 2^m\}^N \\ \|k-l\|=1}} P\left\{\left|X_{t_0+k.2^{-(m+n)}} - X_{t_0+l.2^{-(m+n)}}\right| > 2^{-\tilde{\sigma}(m+n)}2^{s'n}\right\} \\
&\leq \sum_{\substack{k,l\in\{0,\ldots,\pm 2^m\}^N \\ \|k-l\|=1}} \frac{E\left[X_{t_0+k.2^{-(m+n)}} - X_{t_0+l.2^{-(m+n)}}\right]^{2p}}{2^{-2p\tilde{\sigma}(m+n)}2^{2ps'n}} \\
&\leq 2N\; C\; \lambda_p \underbrace{\#\left\{0,\ldots,\pm 2^m\right\}^N}_{\left(1+2^{(m+1)}\right)^N} 2^{-2p\epsilon(m+n)}
\end{aligned}$$

where $\lambda_p$ is the positive constant such that for all centered Gaussian random variable $Y$ and all $p \in \mathbf{N}^*$, we have $E\left[Y^{2p}\right] = \lambda_p E\left[Y^2\right]$.
Then,

$$P\left\{\Omega_n^m\right\} \leq \left(2N.2^{2N}\right) C\; \lambda_p\; 2^{(N-2p\epsilon)m}2^{-2p\epsilon n}$$

Choosing $p \in \mathbf{N}$ such that $N - 2p\epsilon < 0$, we deduce

$$\begin{aligned}
P\left\{\exists m; \Omega_n^m\right\} = P\left\{\bigcup_m \Omega_n^m\right\} &\leq \sum_m P\left\{\Omega_n^m\right\} \\
&\leq \frac{\left(2N.2^{2N}\right) C\; \lambda_p\; 2^{-2p\epsilon n}}{1 - 2^{N-2p\epsilon}}
\end{aligned}$$

The Borel-Cantelli lemma implies existence of a random variable $n^* \geq n_0$ such that, almost surely,

$$\forall n \geq n^*, \forall m \in \mathbf{N}; \max_{\substack{k,l\in\{0,\ldots,\pm 2^m\}^N \\ \|k-l\|=1}} \left|X_{t_0+k.2^{-(m+n)}} - X_{t_0+l.2^{-(m+n)}}\right| \leq 2^{-\tilde{\sigma}(m+n)}2^{s'n}$$

Therefore, by induction, we get for all $n \geq n^*$ and all $m \in \mathbf{N}$

$$\forall q > m,\ \forall s,t \in D_q;\ \text{s.t. } \|t-s\| < 2^{-(m+n)};$$

$$\begin{aligned}
|X_t - X_s| &\leq 2\left(\sum_{j=m+1}^{q} 2^{-\tilde{\sigma}(j+n)}\right)2^{s'n} \\
&\leq \frac{2.2^{-\tilde{\sigma}(m+n+1)}}{1-2^{-\tilde{\sigma}}}2^{s'n}
\end{aligned}$$

which leads to

$$\forall s,t \in D_n(t_0) = \bigcup_m D_n^m(t_0);\ |X_t - X_s| \leq \frac{2}{1-2^{-\tilde{\sigma}}}\|t-s\|^{\tilde{\sigma}}2^{s'n}$$

and using the continuity of $X$,

$$\forall s,t \in B(t_0, 2^{-n});\ |X_t - X_s| \leq \frac{2}{1-2^{-\tilde{\sigma}}}\|t-s\|^{\tilde{\sigma}}2^{s'n}$$

Hence, almost surely, for all $\rho \in (0, 2^{-n^*})$, there exists $n > n^*$ such that $2^{-(n+1)} \leq \rho \leq 2^{-n}$ and

$$\begin{aligned} \forall s,t \in B(t_0,\rho);\ |X_t - X_s| &\leq \frac{2}{1-2^{-\tilde{\sigma}}} \|t-s\|^{\tilde{\sigma}} 2^{s'n} \\ &\leq 2^{-s'} \frac{2}{1-2^{-\tilde{\sigma}}} \|t-s\|^{\tilde{\sigma}} \rho^{-s'} \end{aligned} \tag{38}$$

□

**Remark 4.2** • *The assumptions of proposition 4.1 are equivalent to the existence of $\sigma > 0$ and $s' < 0$ such that*

$$\limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E[X_t - X_s]^2}{\|t-s\|^{2\sigma} \rho^{-2s'}} < +\infty$$

• *The proposition 4.1 states that under the previous assumption, we have*

$$\forall \epsilon > 0;\ P\{\boldsymbol{\sigma}_{t_0}(s') \geq \sigma - \epsilon\} = 1$$

*which implies, taking $\epsilon \in \boldsymbol{Q}_+$*

$$P\{\boldsymbol{\sigma}_{t_0}(s') \geq \sigma\} = 1 \tag{39}$$

**Remark 4.3** *In proposition 4.1, if (37) holds for all $t_0$ in $[a,b]$, by Kolmogorov's criterion, the process $X$ admits a version $\tilde{X}$, which is continuous on $[a,b]$. Therefore, it would seem unnecessary to assume in addition that $X$ is continuous: The regularity of the paths of $X$ would stand for the regularity of the paths of one of its continuous version $\tilde{X}$.*
*However, as we only have*

$$\forall t \in \boldsymbol{R}_+^N;\ P\left\{X_t = \tilde{X}_t\right\} = 1$$

*the question whether the conclusions of proposition 4.1 could hold uniformly in $t_0 \in \boldsymbol{R}_+^N$ raises a problem. The answer would then depend on which continuous version is considered. As all the continuous versions of $X$ are indistinguishable, this topic seems worth further investigation.*

### 4.1.2 Uniform almost sure result on $\mathbf{R}_+^N$

The remark 2.2 shows that an uniform lower bound for the local Hölder exponent gives an uniform lower bound for $\boldsymbol{\sigma}_{t_0}(s')$.

**Proposition 4.4** *Let $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ be a continuous Gaussian process. Assume that there exists a function $\sigma$ which admits a positive lower bound, and that for all $t_0 \in \boldsymbol{R}_+^N$, there exists $C_0 > 0$ and $\rho_0 > 0$ such that*

$$\forall s,t \in B(t_0,\rho_0), E[X_t - X_s]^2 \leq C_0 \|t-s\|^{2\sigma(t_0)} \tag{40}$$

*Then, almost surely*

$$\forall t_0 \in \boldsymbol{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t_0) \geq \liminf_{u \to t_0} \sigma(u) \tag{41}$$

*and, as a consequence*

$$\forall t_0 \in \boldsymbol{R}_+^N, \forall s' < 0;\ \boldsymbol{\sigma}_{t_0}(s') \geq s' + \liminf_{u \to t_0} \sigma(u) \tag{42}$$

**Proof** First, let us suppose that the function $\sigma$ is constant.
By (40) and Kolmogorov's criterion, for all $t_0 \in \mathbf{R}_+^N$, there exists a modification $\tilde{X}_{t_0}$ of $X$ which is $\alpha$-Hölder continuous for all $\alpha \in (0, \sigma)$ on $B(t_0, \rho_0)$. Therefore the local Hölder exponent of $\tilde{X}_{t_0}$ satisfy

$$\forall t \in B(t_0, \rho_0);\ \tilde{\boldsymbol{\alpha}}_{\tilde{X}_{t_0}}(t) \geq \sigma$$

As a consequence, for all $t_0 \in \mathbf{R}_+^N$, there exists $\rho_0 > 0$ such that

$$P\left\{\forall t \in B(t_0, \rho_0);\ \tilde{\boldsymbol{\alpha}}(t) \geq \sigma\right\} = 1$$

For all $a, b \in \mathbf{Q}_+^N$, such that $a \prec b$, we have

$$[a, b] \subset \bigcup_{t_0 \in [a,b]} B(t_0, \rho_0)$$

As $[a, b]$ is compact, there exists a finite number of balls $B_1, \ldots, B_n$ such that

$$[a, b] \subset \bigcup_{i=1}^{n} B_i$$

and

$$\forall i = 1, \ldots, n;\ P\left\{\forall t \in B_i;\ \tilde{\boldsymbol{\alpha}}(t) \geq \sigma\right\} = 1$$

Therefore, we get

$$P\left\{\forall t \in [a, b];\ \tilde{\boldsymbol{\alpha}}(t) \geq \sigma\right\} = 1$$

As $\mathbf{R}_+^N$ can be covered by a countable number of compact sets $[a, b]$, this leads to

$$P\left\{\forall t \in \mathbf{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t) \geq \sigma\right\} = 1 \tag{43}$$

Using (4) and the continuity of $s' \mapsto \boldsymbol{\sigma}_{t_0}(s')$, the result follows.

In the general case of a non-constant function $\sigma$, for all $a, b \in \mathbf{Q}_+^N$ with $a \prec b$ and all $\sigma = \inf_{u \in [a,b]} \sigma(u) - \epsilon$ with $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\forall s, t \in [a, b];\ E\left[X_t - X_s\right]^2 \leq C\, \|t - s\|^{\sigma}$$

Then (43) implies the existence of $\Omega^* \subset \Omega$ such that $P\{\Omega^*\} = 1$ and for all $\omega \in \Omega^*$,

$$\forall a, b \in \mathbf{Q}_+^N, \forall \epsilon \in \mathbf{Q}_+, \forall t_0 \in \overbrace{[a, b]}^{\circ};\ \tilde{\boldsymbol{\alpha}}(t_0) \geq \inf_{u \in [a,b]} \sigma(u) - \epsilon$$

Therefore, taking two sequences $(a_n)_{n \in \mathbf{N}}$ and $(b_n)_{n \in \mathbf{N}}$ such that $\forall n \in \mathbf{N};\ a_n < t_0 < b_n$ and converging to $t_0$, we have for all $\omega \in \Omega^*$

$$\forall t_0 \in \mathbf{R}_+^N;\ \tilde{\boldsymbol{\alpha}}(t_0) \geq \liminf_{u \to t_0} \sigma(u)$$

□

The uniform almost sure lower bound for the 2-microlocal frontier can be improved when the incremental covariance $E\left[X_t - X_s\right]$ in the ball $B(t_0, \rho)$, admits an upper bound function of $\|t - s\|$ and $\rho$, uniformly in $t_0$. Let us state

**Proposition 4.5** *Let* $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ *be a continuous Gaussian process. Assume that for all* $t_0$*, there exists a real function* $\varsigma_{t_0}$ *such that*

$$\forall t_0, \forall s' < 0; \quad \varsigma_{t_0}(s') \geq m(s') > 0$$

*and that there exists two constants* $C > 0$ *and* $\delta > 0$ *such that for all* $t_0 \in [0,1]$ *and all* $s' < 0$

$$\forall 0 < \rho < \delta,\ \forall s,t \in B(t_0,\rho);\ E\left[X_t - X_s\right]^2 \leq C\ \|t-s\|^{2\varsigma_{t_0}(s')}\rho^{-2s'} \tag{44}$$

*Then for all* $s' < 0$*, almost surely*

$$\forall t_0 \in [0,1]; \quad \boldsymbol{\sigma}_{t_0}(s') \geq \liminf_{u \to t_0} \varsigma_u(s')$$

**Proof** First, let us suppose that $s'$ is fixed and that the function $\varsigma$ is constant equal to $\sigma$.
Let $\epsilon > 0$ such that $\tilde{\sigma} = \sigma - \epsilon > 0$. Let us take $\rho = 2^{-n}$ for $n \geq n_0 = -\log_2 \delta$ and set $D_n^m(t_0) = \left\{t_0 + k.2^{-(m+n)}; k \in \{0, \pm 1, \ldots, \pm(2^m - 1)\}^N\right\}$. Let us consider the event

$$\Omega_n^m = \left\{ \max_{\substack{i \in \left\{0,\ldots,2^{m+n}\right\}^N \\ k,l \in \{0,\ldots,\pm 2^m\}^N \\ \|k-l\|=1}} \left|X_{(i+k).2^{-(m+n)}} - X_{(i+l).2^{-(m+n)}}\right| > 2^{-\tilde{\sigma}(m+n)}2^{s'n} \right\}$$

For all $p \in \mathbf{N}^*$, we have

$$\begin{aligned}
P\left\{\Omega_n^m\right\} &\leq \sum_{\substack{i \in \left\{0,\ldots,2^{m+n}\right\}^N \\ k,l \in \{0,\ldots,\pm 2^m\}^N \\ \|k-l\|=1}} P\left\{\left|X_{(i+k).2^{-(m+n)}} - X_{(i+l).2^{-(m+n)}}\right| > 2^{-\tilde{\sigma}(m+n)}2^{s'n}\right\} \\
&\leq \sum_{\substack{i \in \left\{0,\ldots,2^{m+n}\right\}^N \\ k,l \in \{0,\ldots,\pm 2^m\}^N \\ \|k-l\|=1}} \frac{E\left[X_{(i+k).2^{-(m+n)}} - X_{(i+l).2^{-(m+n)}}\right]^{2p}}{2^{-2p\tilde{\sigma}(m+n)}2^{2ps'n}} \\
&\leq 2N\ C\ \lambda_p\ \underbrace{\#\left\{0,\ldots,\pm 2^m\right\}^N}_{(1+2^{m+1})^N}\ \underbrace{\#\left\{0,\ldots,2^{m+n}\right\}^N}_{(1+2^{m+n})^N} 2^{-2p\epsilon(m+n)}
\end{aligned}$$

where $\lambda_p$ is the positive constant such that for all centered Gaussian random variable $Y$ and all $p \in \mathbf{N}^*$, we have $E\left[Y^{2p}\right] = \lambda_p E\left[Y^2\right]$.
Then,

$$P\left\{\Omega_n^m\right\} \leq (2N.2^{3N})\ C\ \lambda_p\ 2^{(2N-2p\epsilon)m}2^{(N-2p\epsilon)n}$$

Choosing $p \in \mathbf{N}$ such that $2N - 2p\epsilon < 0$, we deduce

$$\begin{aligned}
P\left\{\exists m; \Omega_n^m\right\} = P\left\{\bigcup_m \Omega_n^m\right\} &\leq \sum_m P\left\{\Omega_n^m\right\} \\
&\leq \frac{(2N.2^{3N})\ C\ \lambda_p\ 2^{(N-2p\epsilon)n}}{1 - 2^{2N-2p\epsilon}}
\end{aligned}$$

The Borel-Cantelli lemma implies existence of a random variable $n^* \geq n_0$ such that, almost surely,

$$\forall n \geq n^*, \forall m \in \mathbf{N}; \quad \max_{\substack{i \in \{0,\dots,2^{m+n}\}^N \\ k,l \in \{0,\dots,\pm 2^m\}^N \\ \|k-l\|=1}} \left| X_{(i+k).2^{-(m+n)}} - X_{(i+l).2^{-(m+n)}} \right| \leq 2^{-\tilde{\sigma}(m+n)} 2^{s'n} \tag{45}$$

Therefore, setting $E_r = \left\{ i.2^{-r};\ i \in \{0, \dots, 2^r\}^N \right\} \subset [0,1]$, we show that for all $n \geq n^*$ and all $m \in \mathbf{N}$

$$\begin{aligned} \forall q > m,\ \forall t_0 \in E_{q+n}; \qquad & \forall s,t \in D_n^q(t_0) \text{ s.t. } \|t-s\| < 2^{-(m+n)}; \\ |X_t - X_s| \quad \leq \quad & 2 \left( \sum_{j=m+1}^{q} 2^{-\tilde{\sigma}(j+n)} \right) 2^{s'n} \\ \leq \quad & \frac{2.2^{-\tilde{\sigma}(m+n+1)}}{1-2^{-\tilde{\sigma}}} 2^{s'n} \end{aligned} \tag{46}$$

To prove (46), we proceed by induction:

- for $q = m+1$, for all $t_0 \in E_{m+n+1}$, the conditions $s,t \in D_n^{m+1}(t_0)$ and $\|t-s\| < 2^{-(m+n)}$ impose on $s$ and $t$ to be neighbors in $D_n^{m+1}(t_0)$. Therefore (46) follows from (45).

- assume that the property is valid for an integer $M > m$, then take $t_0 \in E_{M+n+1}$, and $s,t \in D_n^{M+1}(t_0)$ such that $|t-s| < 2^{-(m+n)}$. There exists $\tilde{t_0} \in E_{M+n}$ such that $|t_0 - \tilde{t_0}| \leq 2^{-(M+n+1)}$. As $\tilde{t_0}$ can be chosen such that the following strict inequality holds

$$\begin{aligned} \|s - \tilde{t_0}\| \quad < \quad & \|s - t_0\| + \|t_0 - \tilde{t_0}\| \\ < \quad & 2^{-n} - 2^{-(M+n+1)} + 2^{-(M+n+1)} \end{aligned}$$

  there exists $\tilde{s} \in D_n^M(\tilde{t_0})$ such that $|s - \tilde{s}| \leq 2^{-(M+n+1)}$. In the same way, we get $\tilde{t} \in D_n^M(\tilde{t_0})$ such that $|t - \tilde{t}| \leq 2^{-(M+n+1)}$. Moreover, $\tilde{s}$ and $\tilde{t}$ can be chosen such that $|\tilde{t} - \tilde{s}| \leq |t - s| < 2^{-(m+n)}$. Then, by the triangular inequality

$$|X_t - X_s| \leq |X_t - X_{\tilde{t}}| + |X_{\tilde{t}} - X_{\tilde{s}}| + |X_{\tilde{s}} - X_s|$$

  and the fact that $s, \tilde{s}, \tilde{t}, t$ belong to $D_n^{M+1}(\tilde{t_0})$, (45) gives

$$|X_t - X_s| \leq 2.2^{-\tilde{\sigma}(M+n+1)} 2^{s'n} + |X_{\tilde{t}} - X_{\tilde{s}}|$$

Then property (46) follows.

Let us take $t_0 \in \bigcup_q E_{q+n}$ and $s,t \in \bigcup_q D_n^q(t_0)$. There exists $m > 0$ such that $2^{-(m+n+1)} \leq \|t-s\| < 2^{-(m+n)}$. Then property (46) applied to $m$, $t_0$, $s$ and $t$ gives $|X_t - X_s| \leq \frac{2}{1-2^{-\tilde{\sigma}}} \|t-s\|^{\tilde{\sigma}}.2^{s'n}$.
Using the continuity of $X$, we get

$$\forall t_0 \in [0,1];\ \forall s,t \in B(t_0, 2^{-n});\ |X_t - X_s| \leq \frac{2}{1-2^{-\tilde{\sigma}}} \|t-s\|^{\tilde{\sigma}} 2^{s'n}$$

Hence, almost surely, for all $\rho \in (0, 2^{-n^*})$, there exists $n > n^*$ such that $2^{-(n+1)} \leq \rho \leq 2^{-n}$ and

$$\begin{aligned} \forall s,t \in B(t_0,\rho);\ |X_t - X_s| &\leq \frac{2}{1-2^{-\tilde{\sigma}}}\|t-s\|^{\tilde{\sigma}} 2^{s'n} \\ &\leq 2^{-s'}\frac{2}{1-2^{-\tilde{\sigma}}}\|t-s\|^{\tilde{\sigma}}\rho^{-s'} \end{aligned} \tag{47}$$

In the general case where $\varsigma$ is not constant, for all $a, b \in \mathbf{Q}_+^N$ with $a \prec b$ let us consider $\sigma = \inf_{u\in[a,b]} \varsigma_u(s') - \epsilon$ with $\epsilon > 0$. By (47), there exists a set $\Omega^* \subset \Omega$ such that $P\{\Omega^*\} = 1$ and for all $\omega \in \Omega^*$,

$$\forall a,b \in \mathbf{Q}_+^N, \forall \epsilon \in \mathbf{Q}_+, \forall t_0 \in \overset{\circ}{\overbrace{[a,b]}};\ \boldsymbol{\sigma}(t_0) \geq \inf_{u\in[a,b]} \varsigma_u(s') - \epsilon$$

Therefore, taking two sequences $(a_n)_{n\in\mathbf{N}}$ and $(b_n)_{n\in\mathbf{N}}$ such that $\forall n \in \mathbf{N};\ a_n < t_0 < b_n$ and converging to $t_0$, we have for all $\omega \in \Omega^*$

$$\forall t_0 \in \mathbf{R}_+^N;\ \boldsymbol{\sigma}_{t_0}(s') \geq \liminf_{u\to t_0} \varsigma_u(s') \tag{48}$$

□

## 4.2 Upper bound for the 2-microlocal frontier

### 4.2.1 Pointwise almost sure result

To get the almost sure upper bound for $\boldsymbol{\sigma}_{t_0}(s')$, we need the following

**Lemma 4.6** *Let $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ be a continuous Gaussian process. Assume for some $t_0 \in \boldsymbol{R}_+^N$, there exists $\sigma > 0$ and $s' < 0$ such that there are two sequences $(s_n)_{n\in\boldsymbol{N}}$ and $(t_n)_{n\in\boldsymbol{N}}$ converging to $t_0$, and a constant $c > 0$ such that*

$$\forall n \in \boldsymbol{N};\ E\left[X_{t_n} - X_{s_n}\right]^2 \geq c\|t_n - s_n\|^{2\sigma}\rho_n^{-2s'}$$

*where $\rho_n \geq \max(\|s_n - t_0\|, \|t_n - t_0\|)$. Then the 2-microlocal exponent satisfies almost surely*

$$\boldsymbol{\sigma}_{t_0}(s') \leq \sigma$$

**Proof** Let $\epsilon > 0$ and consider two sequences $(s_n)_{n\in\mathbf{N}}$ and $(t_n)_{n\in\mathbf{N}}$ as in the statement.
For all $n \in \mathbf{N}$, the law of the random variable $\frac{X_{t_n}-X_{s_n}}{\|t_n-s_n\|^{\sigma+\epsilon}\rho_n^{-s'}}$ is $\mathcal{N}(0,\sigma_n^2)$.
From the assumption, we have $\sigma_n \geq c\|t_n - s_n\|^{-2\epsilon} \to +\infty$ as $n \to +\infty$.
Then, for all $\lambda > 0$,

$$\begin{aligned} P\left\{\frac{\|t_n - s_n\|^{\sigma+\epsilon}\rho_n^{-s'}}{|X_{t_n} - X_{s_n}|} < \lambda\right\} &= P\left\{\frac{|X_{t_n} - X_{s_n}|}{\|t_n - s_n\|^{\sigma+\epsilon}\rho_n^{-s'}} > \frac{1}{\lambda}\right\} \\ &= \int_{|x|>\frac{1}{\lambda}} \frac{1}{\sqrt{2\pi}\sigma_n}\exp\left(-\frac{x^2}{2\sigma_n^2}\right).dx \\ &= \frac{1}{2\pi}\int_{|x|>\frac{1}{\lambda\sigma_n}} \exp\left(-\frac{x^2}{2}\right).dx \overset{n\to+\infty}{\longrightarrow} 1 \end{aligned}$$

Therefore the sequence $\left(\frac{\|t_n - s_n\|^{\sigma+\epsilon}\rho_n^{-s'}}{|X_{t_n} - X_{s_n}|}\right)_{n\in\mathbf{N}}$ converge to 0 in probability. This implies existence of a subsequence which converge to 0 almost surely. Then we have almost surely $\boldsymbol{\sigma}_{t_0}(s') \leq \sigma + \epsilon$. Taking $\epsilon \in \mathbf{Q}_+$, the result follows. □

**Remark 4.7** *The assumptions of lemma 4.6 are equivalent to the existence of $\sigma > 0$ and $s' < 0$ such that*

$$\limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\sigma}\rho^{-2s'}} > 0$$

### 4.2.2 Uniform almost sure result on $\mathbf{R}_+^N$

If the assumptions of lemma 4.6 are satisfied for all $t_0$, the conclusion holds uniformly in $t_0$.

**Proposition 4.8** *Let $X = \left\{X_t; t \in \boldsymbol{R}_+^N\right\}$ be a continuous Gaussian process. Suppose that the assumptions of lemma 4.6 are satisfied for all $t_0 \in \boldsymbol{R}_+^N$, with the same $\sigma$ and $s'$. Then the 2-microlocal exponent satisfies almost surely*

$$\forall t_0 \in \boldsymbol{R}_+^N;\ \boldsymbol{\sigma}_{t_0}(s') \leq \sigma$$

**Proof** First of all, in a classical way, lemma 4.6 imply

$$\forall \tilde{\sigma} > \sigma;\ P\left\{\forall t_0 \in \mathbf{Q}_+^N; \boldsymbol{\sigma}_{t_0}(s') < \tilde{\sigma}\right\} = 1$$

To extend this result for all $t_0 \in \mathbf{R}_+^N$, let us consider $t_0 \in \mathbf{R}_+^N - \mathbf{Q}_+^N$, and a sequence $(x^{(m)})_{m\in\mathbf{N}}$ in $\mathbf{Q}_+^N$ such that $x^{(m)} \to t_0$. We have, almost surely, for all $m$, $\boldsymbol{\sigma}_{x^{(m)}}(s') < \tilde{\sigma}$. Then there exists sequences $(\rho_n^{(m)})_{n\in\mathbf{N}}$, $(s_n^{(m)})_{n\in\mathbf{N}}$ and $(t_n^{(m)})_{n\in\mathbf{N}}$ such that $\forall n; s_n^{(m)}, t_n^{(m)} \in B(x^{(m)}, \rho_n^{(m)})$ and

$$\lim_{n\to\infty} \frac{|X_{t_n^{(m)}} - X_{s_n^{(m)}}|}{\|t_n^{(m)} - s_n^{(m)}\|^{\tilde{\sigma}}(\rho_n^{(m)})^{-s'}} = +\infty \tag{49}$$

From these $m$ sequences, we build 3 sequences $(s_n)$, $(t_n)$ and $(\rho_n)$ such that $s_n \to t_0$, $t_n \to t_0$, $s_n, t_n \in B(t_0, \rho_n)$ and

$$\lim_{n\to\infty} \frac{|X_{t_n} - X_{s_n}|}{\|t_n - s_n\|^{\tilde{\sigma}}(\rho_n)^{-s'}} = +\infty$$

For all $n$ and $m$, let us write

$$\left\{\begin{array}{l} s_n^{(m)} - t_0 = s_n^{(m)} - x^{(m)} + x^{(m)} - t_0 \\ t_n^{(m)} - t_0 = t_n^{(m)} - x^{(m)} + x^{(m)} - t_0 \end{array}\right. \tag{50}$$

Let us fix $\epsilon > 0$, there exists $N > 0$ such that

$$\forall m \geq N, |x^{(m)} - t_0| < \epsilon \tag{51}$$

As for all $m$, $|s_n^{(m)} - x^{(m)}| \to 0$ when $n \to \infty$, there exists a subsequence $(s_{p_n}^{(m)})_n$ such that

$$\forall n;\ |s_{p_n}^{(m)} - x^{(m)}| < |s_n^{(m-1)} - x^{(m-1)}|$$

therefore, we can suppose that the sequences $(s_n^{(m)})_n$ satisfy the properties

$$\forall m,n;\ |s_n^{(m)} - x^{(m)}| < |s_n^{(m-1)} - x^{(m-1)}| \tag{52}$$

Moreover, in the same way, we can suppose

$$\forall m,n;\ |t_n^{(m)} - x^{(m)}| < |t_n^{(m-1)} - x^{(m-1)}| \tag{53}$$

Then, using the fact that there exists $N' \in \mathbf{N}$ such that

$$\forall n \geq N';\ \begin{cases} |s_n^{(1)} - x^{(1)}| < \epsilon \\ |t_n^{(1)} - x^{(1)}| < \epsilon \end{cases}$$

(52) and (53) imply

$$\forall n \geq N';\ \begin{cases} |s_n^{(n)} - x^{(n)}| < \epsilon \\ |t_n^{(n)} - x^{(n)}| < \epsilon \end{cases} \tag{54}$$

Therefore (50), (51) and (54) lead to

$$\lim s_n^{(n)} = \lim t_n^{(n)} = t_0 \tag{55}$$

In the same way than previously, by (49), we can suppose that the sequences $(s_n^{(m)})_n$, $(t_n^{(m)})_n$ and $(\rho_n^{(m)})_n$ satisfy

$$\forall m,n;\ \frac{|X_{t_n^{(m)}} - X_{s_n^{(m)}}|}{\|t_n^{(m)} - s_n^{(m)}\|^{\tilde{\sigma}} (\rho_n^{(m)})^{-s'}} > \frac{|X_{t_n^{(m-1)}} - X_{s_n^{(m-1)}}|}{\|t_n^{(m-1)} - s_n^{(m-1)}\|^{\tilde{\sigma}} (\rho_n^{(m-1)})^{-s'}} \tag{56}$$

Then, for all $M > 0$, there exists $N'' \in \mathbf{N}$ such that

$$\forall n \geq N'';\ \frac{|X_{t_n^{(1)}} - X_{s_n^{(1)}}|}{\|t_n^{(1)} - s_n^{(1)}\|^{\tilde{\sigma}} (\rho_n^{(1)})^{-s'}} > M$$

which leads to, using (56)

$$\forall n \geq N'';\ \frac{|X_{t_n^{(n)}} - X_{s_n^{(n)}}|}{\|t_n^{(n)} - s_n^{(n)}\|^{\tilde{\sigma}} (\rho_n^{(n)})^{-s'}} > M \tag{57}$$

We have shown

$$\lim_{n \to \infty} \frac{|X_{t_n^{(n)}} - X_{s_n^{(n)}}|}{\|t_n^{(n)} - s_n^{(n)}\|^{\tilde{\sigma}} (\rho_n^{(n)})^{-s'}} = +\infty \tag{58}$$

Therefore (55) and (58) imply $\boldsymbol{\sigma}_{t_0}(s') \leq \tilde{\sigma}$. Then we can state that, almost surely,

$$\forall t_0 \in \mathbf{R}_+^N; \boldsymbol{\sigma}_{t_0}(s') \leq \sigma$$

□

**Theorem 4.9** *Let* $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ *be a continuous Gaussian process. Assume that for all* $t_0 \in \boldsymbol{R}_+^N$ *and all* $s' < 0$*, there exists* $\varsigma_{t_0}(s') > 0$ *such that*

$$\limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\varsigma_{t_0}(s')}\rho^{-2s'}} > 0$$

*Then for all* $s' < 0$*, we have almost surely*

$$\forall t_0 \in \boldsymbol{R}_+^N;\ \boldsymbol{\sigma}_{t_0}(s') \leq \limsup_{u \to t_0} \varsigma_u(s')$$

**Proof** Under the assumptions of the theorem, for all $a, b \in \mathbf{Q}_+^N$ with $a \prec b$ and all $\varsigma(s') = \sup_{u \in [a,b]} \varsigma_u(s')$,

$$\forall t_0 \in [a,b];\ \limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{E\left[X_t - X_s\right]^2}{\|t-s\|^{2\varsigma(s')}\rho^{-2s'}} > 0$$

Therefore, by proposition 4.8, for all $s' < 0$, there exists a set $\Omega^* \subset \Omega$ with $P\left\{\Omega^*\right\} = 1$ such that for all $\omega \in \Omega^*$

$$\forall a, b \in \mathbf{Q}_+^N, \forall t_0 \in \overbrace{[a,b]}^{\circ};\ \boldsymbol{\sigma}_{t_0}(s') \leq \sup_{u \in [a,b]} \varsigma_u(s')$$

Taking two sequences $(a_n)_{n \in \mathbf{N}}$ and $(b_n)_{n \in \mathbf{N}}$ such that $\forall n \in \mathbf{N};\ a_n < t_0 < b_n$ and converging to $t_0$, we have for all $\omega \in \Omega^*$

$$\forall t_0 \in \mathbf{R}_+^N;\ \boldsymbol{\sigma}_{t_0}(s') \leq \limsup_{u \to t_0} \varsigma_u(s')$$

□

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## Annexe au chapitre IV
## 0 − 1 laws

A stochastic process $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ is said to belong to $C_{t_0}^{\sigma,s'}$ where $t_0 \in \mathbf{R}_+^N$, $0 < \sigma \leq 1$, $s' < 0$ and $\sigma - s' > 0$, if

$$\limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} < \infty \tag{IV.1}$$

The 2-microlocal frontier of $X$ at $t_0$ is defined to be the random function $s' \mapsto \boldsymbol{\sigma}_{t_0}(s') = \inf\left\{\sigma :\ X \in C_{t_0}^{\sigma,s'}\right\}$.
We can see easily that (IV.1) is equivalent to the existence of $C > 0$ and $\rho_0 > 0$ such that

$$\forall 0 < \rho < \rho_0; \forall s,t \in B(t_0,\rho); \quad |X_t - X_s| \leq C\|t-s\|^\sigma \rho^{-s'} \tag{IV.2}$$

In order to get an almost sure lower bound for the 2-microlocal frontier of a Gaussian process uniformly on $\mathcal{T} \subset \mathbf{R}_+^N$, a useful result about zero-one laws for Gaussian processes can be used (see [1])

If $\Lambda$ is a set of real functions on $\mathcal{T}$ which contains the paths of the Gaussian process $X$, let denote $\mathcal{B}(\Lambda)$ the $\sigma$-algebra of subsets of $\Lambda$ generated by sets of the form

$$\{f \in \Lambda : (f(t_1), \dots, f(t_n)) \in B\}$$

where $t_1, \dots, t_n \in \mathcal{T}$ and $B$ is a Borel set of $\mathbf{R}^n$.
Consider the transformation $\phi : (\Omega, \mathcal{F}, P) \to (\Lambda, \mathcal{B}(\Lambda))$

$$\phi(\omega) = \begin{cases} X_\bullet(\omega) & \text{if } X_\bullet(\omega) \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

The function $\phi$ is measurable and induces a probability measure $\mu$ on $(\Lambda, \mathcal{B}(\Lambda))$. Let us denote by $\bar{\mathcal{B}}(\Lambda)$ the completion of $\mathcal{B}(\Lambda)$ with respect to $\mu$ and by $H(R)$ the reproducing kernel Hilbert space determined by the covariance $R$. Assume the two following conditions:

(C1) $\Lambda$ is a linear space;

(C2) $H(R) \subset \Lambda$.

Both conditions (C1) and (C2) are satisfied in each of the following cases:

1. $\Lambda = \mathbf{R}^{\mathcal{T}}$,
2. $\mathcal{T}$ is a measurable subset of $\mathbf{R}$, the process $X$ is measurable and $\Lambda$ is the set of real measurable functions on $\mathcal{T}$,
3. $\mathcal{T}$ is a metric space, $X$ is a.s. continuous and $\Lambda = C(\mathcal{T})$.

**Theorem 0.1** *Let $\Lambda$ be a set of real functions on $\mathcal{T}$ which contains the paths of $X$ [P]-a.s., and satisfy conditions (C1) and (C2). If $F \in \bar{\mathcal{F}}$ is such that $F = \phi^{-1}(G)$, where $G$ is a $\bar{\mathcal{B}}(\Lambda)$-measurable subgroup of $\Lambda$, then $P\{F\} = 0$ or 1.*

In [1], theorem 0.1 is used to show some results which occur with probability 0 or 1, for a real Gaussian separable process (e.g. boundness on $\mathcal{T}$, continuity on $\mathcal{T}$, Lipschitz condition of order $\alpha > 0$ on $\mathcal{T}$, ...).

Recall that a stochastic process $X = \{X_t;\ t \in \mathcal{T}\}$, where $\mathcal{T} \subset \mathbf{R}_+^N$, is said to be separable if there exists an at most countable dense collection $\mathcal{S} \subset \mathcal{T}$, and a null set $\Omega_0$ such that for all closed set $A \subset \mathbf{R}$ and all open set $I \subset \mathcal{T}$ of the form $I = \prod_{i=1}^N ]\alpha^{(l)}; \beta^{(l)}[$, where $\alpha^{(l)}, \beta^{(l)} \in \overline{\mathbf{Q}}$,

$$\{\omega : X_s(\omega) \in A;\ \forall s \in I \cap \mathcal{S}\} \setminus \{\omega : X_s(\omega) \in A;\ \forall s \in I\} \subset \Omega_0$$

In [3], it is shown that this property is equivalent to: for all $\omega \in \Omega \setminus \Omega_0$ and all $t \in \mathcal{T} \setminus \mathcal{S}$, there exists a sequence $(s_n)_{n \in \mathbf{N}}$ in $\mathcal{S}$ converging to $t$ such that $\lim_{n\to\infty} X_{s_n}(\omega) = X_t(\omega)$.

By Doob's separability theorem, any stochastic process $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ has a separable modification (see [2]).

**Remark 0.2** • *A continuous process $X = \left\{X_t;\ t \in \boldsymbol{R}_+^N\right\}$ is separable.*

*Let $\Omega_0 = \{\omega : X_\bullet(\omega) \text{ not continuous}\}$ and $\mathcal{S} = \boldsymbol{Q}_+^N$, by the last characterisation, we can see that $X$ is separable.*

- *Two different separable modifications of a stochastic process are indistinguishable.*

  *This fact is a simple consequence of the second characterisation of separability.*

Assume that $X$ is a separable real Gaussian process, indexed by $\mathbf{R}_+^N$.
Let us consider the event

$$\begin{aligned} F &= \left\{\omega :\ \forall t_0 \in \mathcal{T};\ \limsup_{\rho\to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} < \infty\right\} \\ &= \left\{\omega :\ \forall t_0 \in \mathcal{T};\ \sup_{\rho>0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} < \infty\right\} \end{aligned}$$

Our goal is to show that $P\{F\} = 0$ or 1.

To study measurability, it is convenient to express $F$ as

$$\begin{aligned} F &= \bigcap_{t_0\in\mathcal{T}} \bigcup_{n\in\mathbf{N}^*} \bigcap_{\rho>0} \left\{\omega :\ \sup_{s,t\in B(t_0,\rho)} \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} < n\right\} \\ &= \bigcap_{t_0\in\mathcal{T}} \bigcup_{n\in\mathbf{N}^*} \bigcap_{\rho>0} \bigcap_{s\in B(t_0,\rho)} \bigcap_{t\in B(t_0,\rho)} \left\{\omega :\ \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} < n\right\} \end{aligned} \tag{IV.3}$$

Then, let us consider

$$F' = \bigcap_{t_0\in\mathcal{S}} \bigcup_{n\in\mathbf{N}^*} \bigcap_{\rho\in\mathbf{Q}_+^*} \bigcap_{s\in B(t_0,\rho)\cap\mathcal{S}} \bigcap_{t\in B(t_0,\rho)\cap\mathcal{S}} \left\{\omega :\ \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} < n\right\}$$

It can be easily seen that $F' \in \mathcal{F}$.
Taking $\Lambda = \mathbf{R}^{\mathcal{T}}$, the set

$$G' = \left\{f \in \Lambda :\ \forall t_0 \in \mathcal{S};\ \limsup_{\substack{\rho\to 0\\ \rho\in\mathbf{Q}}} \sup_{s,t\in B(t_0,\rho)\cap\mathcal{S}} \frac{|f(t) - f(s)|}{\|t-s\|^\sigma \rho^{-s'}} < \infty\right\}$$

is a group and $\mathcal{B}(\Lambda)$-measurable. Moreover $F' = \phi^{-1}(G')$.
Then, theorem 0.1 implies $P\{F'\} = 0$ or 1.

We have to show that $F' \setminus F \subset \Omega_0$ (we can see easily that $F \subset F'$), which implies $P\{F\} = P\{F'\}$. As

$$F' \setminus F \subset \Omega_0 \Leftrightarrow \forall \omega \in F' \setminus \Omega_0;\ \omega \in F$$

Then, let $\omega \in F' \setminus \Omega_0$,

$$\forall t_0 \in \mathcal{S}, \exists n \in \mathbf{N}, \forall \rho \in \mathbf{Q}_+^*, \forall s,t \in B(t_0,\rho) \cap \mathcal{S};\quad \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} \leq n$$

Suppose $t_0, n, \rho$ fixed, if $s,t \in \mathcal{T} \backslash \mathcal{S}$, consider sequences $\left(s^{(i)}\right)_{i\in\mathbf{N}}$ and $\left(t^{(i)}\right)_{i\in\mathbf{N}}$ in $\mathcal{S}$, converging to $s$ and $t$ and such that $\lim_{i\to\infty} X_{s^{(i)}}(\omega) = X_s(\omega)$ and $\lim_{i\to\infty} X_{t^{(i)}}(\omega) = X_t(\omega)$.
As for all $i$, $|X_{t^{(i)}} - X_{s^{(i)}}| \leq n\ \|t^{(i)} - s^{(i)}\|^\sigma \rho^{-s'}$, making $i \to \infty$, we get $|X_t - X_s| \leq n\ \|t-s\|^\sigma \rho^{-s'}$, and then

$$\forall t_0 \in \mathcal{S}, \exists n \in \mathbf{N}, \forall \rho > 0, \forall s,t \in B(t_0,\rho);\quad \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} \leq n$$

Therefore, we have

$$\forall \sigma > 0, \forall s' < 0;\quad P\left\{\forall t_0 \in \mathcal{S};\ \boldsymbol{\sigma}_{t_0}(s') \geq \sigma\right\} = 0 \text{ or } 1 \tag{IV.4}$$

To get an almost sure result for $t_0 \in \mathcal{T}$, we need an uniform boundness of $\frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}}$.
However, we can see that such an assumption gives a result which can be reached immediately. Indeed, taking

$$F = \bigcup_{n\in\mathbf{N}^*} \bigcap_{t_0\in\mathcal{T}} \bigcap_{\rho>0} \bigcap_{s\in B(t_0,\rho)} \bigcap_{t\in B(t_0,\rho)} \left\{\omega :\ \frac{|X_t - X_s|}{\|t-s\|^\sigma \rho^{-s'}} < n\right\}$$

instead of (IV.3), we can show directly that

$$P\left\{\exists C > 0, \forall t_0 \in \mathcal{T}, \forall \rho > 0, \forall s,t \in B(t_0,\rho);\ |X_t - X_s| \leq C\|t-s\|^\sigma \rho^{-s'}\right\} = 0 \text{ or } 1$$

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**Résumé**

Le mouvement brownien multifractionnaire est une généralisation du bien connu mouvement brownien fractionnaire, pour lequel le paramètre d'autosimilarité est remplacé par une fonction. Cette substitution permet à la régularité locale de varier le long des trajectoires.

Dans un premier article, des extensions de ces processus pour des paramètres multidimensionnels, sont étudiées. Pour chacun d'eux, deux types d'extension sont définis : l'une est isotrope, l'autre non. Les propriétés fractales des processus à paramètre réel, sont étendues : l'autosimilarité et la stationnarité des accroissements dans le cas fractionnaire, et l'autosimilarité asymptotique locale dans le cas multifractionnaire.

La construction et l'étude d'un mouvement brownien fractionnaire indexé par des ensembles, est l'objet d'un deuxième article. Les propriétés fractales sont définies pour un processus indexé par des ensembles et prouvées dans le cas de notre processus. Enfin, son comportement sur les chemins croissants est examiné : le processus à paramètre réel obtenu par projection sur les flots, est un fBm classique changé de temps.

Dans un troisième article, on étend l'analyse 2-microlocale au cadre stochastique des processus gaussiens. Celle-ci permet de prédire la régularité des processus obtenus par action d'opérateurs intégro-différentiels. La valeur presque sure de la frontière 2-microlocale du mouvement brownien (multi-)fractionnaire est évaluée.

**Mots-clefs** autosimilarité, autosimilarité asymptotique locale, mouvement brownien fractionnaire, processus à plusieurs paramètres, processus gaussiens, processus indexés par des ensembles, régularité höldérienne, stationnarité.

**Abstract**

The multifractional Brownian motion is a generalization of the well-known fractional Brownian motion, where the constant index of self-similarity is substitued with a function. This substitution allows the local regularity to vary along the paths. In a first paper, multiparameter extensions of these processes are studied. For each of one, two kinds of extension are defined : one is isotropic, the other is not. The fractal properties of one-parameter processes are extended : self-similarity and increments stationarity in the fractional case, and the locally asymptotic self-similarity in the multifractional case.

The definition and study of a set-indexed fractional Brownian motion, is the object of a second paper. Fractal properties are defined for a set-indexed process, and proved for our process. Eventually, behavior along increasing paths are examined : the one-parameter process obtained from projection along flows, is a time-changed classical fBm.

In a third paper, we extend the 2-microlocal analysis to the stochastic case of Gaussian processes. This allows to predict the regularity of processes obtained by action of integro-differential operators. The almost sure value of the 2-microlocal frontier of the (multi-)fractional Brownian motion is computed.

**Keywords** fractional Brownian motion, Gaussian processes, Hölder regularity, local asymptotic self-similarity, multiparameter processes, stationarity, self-similarity, set-indexed processes.

**AMS classification** 62 G 05, 60 G 15, 60 G 17, 60 G 18.